A proof of the nonexistence of a function from $\mathbb{R}_+^2$ to $\mathbb{R}_+$ which is strictly monotonic

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In this article, we will prove there does not exist a function f: $\mathbb{R}_+^2 \to \mathbb{R}_+$ such that the value function on the any line of $\mathbb{R}_+^2$ is strictly monotonic.

Proof.

We will prove by contradition. Suppose $\exists f$ satisfies the condition. Without loss of generality, we assume f is strictly incresing on the line $x = 1$ when $y \to \infty$.

Lemma 1. If the Lebesgue measure of every element in an uncountable set S is greater than zero, then the summation of measure of every element in S is ∞ .

Proof. Suppose summation of measure of every element is finite.

Let $A_n = \{s \in S | \mu(s) > \frac{1}{n}\}$. Then $\forall n$, A_n must have a finite number of elements (or the sum will go to ∞). Notice $S = \bigcup_{n=1}^{\infty} A_n$ is countable, which contradicts the condition. So Lemma 1 is proved by contradition.

Lemma 2. $\forall X_0 = (x_0, y_0) \in R_+^2$, $\exists$ a line $l_{X_0}((x_0, y_0))$ such that

(1) $l_{X_0}((x_0, y_0)) = 0$.

(2)$\forall (x_1, y_1), (x_2, y_2)$ s.t. $f((x_1, y_1)) > f((x_0, y_0)) > f((x_2, y_2))$, we have $l_{X_0}((x_1, y_1)) > l_{X_0}((x_0, y_0)) > l_{X_0}((x_2, y_2))$

Proof. Suppose the statement is not true. Given $X_0 = (x_0, y_0)$,there exists three rays l_1, l_2, l_3 in clockwise direction in a halfplane, which X_0 is their common vertices. Without lost of generality, we can assume f strictly increases through l_1 and l_3 but strictly decreases through l_2. Since l_1, l_2, l_3 are in a halfplane, we can pick $X_i (X_i \neq X_0)$ on the ray $l_i (i = 1, 2, 3;)$ which $X_i (i = 1, 2, 3)$ are collinear. Therefore, we have $f(X_1) > f(X_0), f(X_3) > f(X_0)$ but $f(X_2) < f(X_0)$. However, since rays l_1, l_2, l_3 are in clockwise direction and f is strictly monotonic on the line decided by point X_1, X_2, X_3,

$f(X_2) > \min\{f(X_1), f(X_3))\}$ This is a contradiction.

Lemma 3. $\forall X_1 \neq X_2 \in R^2_+$ subject to $l_{X_1}(X_2) \neq 0$ & $l_{X_2}(X_1) \neq 0$, l_{X_1} and l_{X_2} does not intersect.

Proof. Suppose not. $\exists X_1 \neq X_2 \in R^2_+$, $l_{X_1} \cap l_{X_2} = X^*$. Then $\exists X^{**}$ subject to $l_{X_1}(X^{**}) l_{X_2}(X^{**}) < 0$. Hence according to **Lemma 2**, $(f(X^{**}) - f(X^*))^2 < 0$ which is impossible.

Now we focus on the range of function value on l_{X_t} where point $X_t = (1, t)$.

Lemma 4. We define $M_t = \sup\{f(\hat{X}) : \hat{X} \in l_{X_t}\}$ and $m_t = \inf\{f(\hat{X}) : \hat{X} \in l_{X_t}\}$. Notice $M_t - m_t > 0$. We will prove $\forall t_1 > t_2 : m_{t_1} \geq M_{t_2}$

Proof. Suppose not. $\exists t_1 > t_2$ & $X_1 \in l_{X_{t_1}}, X_2 \in l_{X_{t_2}} : f(X_1) < f(X_2)$. By **Lemma 2** & **Lemma 3** and the assuption that f is strictly increasing on the line $x = 1$ when $y \to \infty$, $f(X_1) > f(X_2)$. This contradiction proves this statement.

Now we shall prove the orginial statement.

Define $S = \sum_{t \in [1,2]} (M_t - m_t)$. Notice $M_t - m_t > 0, \forall t \in [0, 1]$. According to **Lemma 1**, $S = \infty$. But according to **Lemma 4**, $f(1, 3) \geq m_3 \geq S + f(1, 0.9) = \infty$. This contradiction proves the original statement. **QED.**