

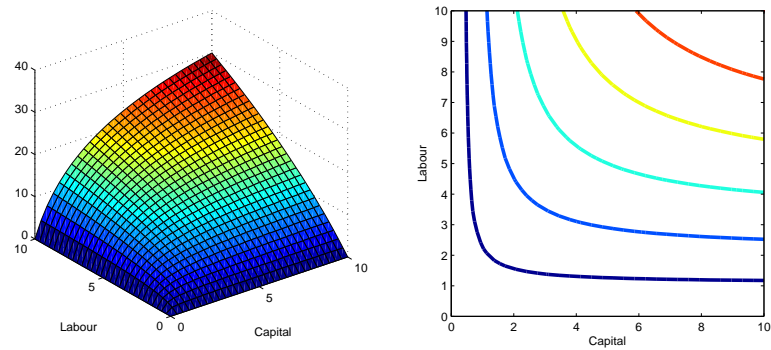
CES functions and Dixit-Stiglitz Formulation

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Abstract

Constant elasticity of substitution (CES) functions are the most extensively used functional form in economics so far, but textbooks seldom give good and enough illustrations on how to use them. The purpose of this note is to show you how it can be derived and its contribution to Dixit-Stiglitz formulation.

1 CES functions

We start with the most basic CES function,

$$u(x_1, x_2) = (x_1^\sigma + x_2^\sigma)^{\frac{1}{\sigma}}$$

where $\sigma < 1$. Let us first verify this is a CES function. Take partial derivatives w.r.t. both x_i 's,

$$\frac{\partial u(x_1, x_2)}{\partial x_1} = \frac{1}{\sigma} (x_1^\sigma + x_2^\sigma)^{\frac{1}{\sigma}-1} \sigma x_1^{\sigma-1} = (x_1^\sigma + x_2^\sigma)^{\frac{1-\sigma}{\sigma}} x_1^{\sigma-1} \quad (1)$$

$$\frac{\partial u(x_1, x_2)}{\partial x_2} = \frac{1}{\sigma} (x_1^\sigma + x_2^\sigma)^{\frac{1}{\sigma}-1} \sigma x_2^{\sigma-1} = (x_1^\sigma + x_2^\sigma)^{\frac{1-\sigma}{\sigma}} x_2^{\sigma-1} \quad (2)$$

Then recall how we define *price elasticity of demand*,

$$\varepsilon^p = \frac{\Delta x/x}{\Delta p/p}$$

Most of time, we work with continuous case,

$$\varepsilon^p = \frac{dx/x}{dp/p} = \frac{dx}{dp} \frac{p}{x}$$

We assume only one good in this case.

The *price elasticity of substitution* looks similar, but a little bit complicated, since it is about substitution, we need at least two goods, say x_1 and x_2 . This is a very important concept in comparative statics, defined as: *the relative change of the ratio of the consumption goods x_1 and x_2 over the relative change of the ratio of the according price p_1 and p_2 ,*

$$\varepsilon^{sub} = \frac{\frac{d(x_1/x_2)}{x_1/x_2}}{\frac{d(p_2/p_1)}{p_2/p_1}}$$

We need to show the case at the optimum, thus the point-elasticity at the optimum is,

$$\varepsilon^{sub} = \frac{\frac{d(x_1^*/x_2^*)}{x_1^*/x_2^*}}{\frac{d(p_2/p_1)}{p_2/p_1}} = \frac{\frac{d(x_1^*/x_2^*)}{x_1^*/x_2^*}}{\frac{d(u_2/u_1)}{u_2/u_1}} = \frac{d\left(\frac{x_1^*}{x_2^*}\right)}{d\left(\frac{p_2}{p_1}\right)} \cdot \frac{p_2}{p_1} \cdot \frac{x_1^*}{x_2^*} \quad (3)$$

where $u_1 = \partial u / \partial x_1^*$, $u_2 = \partial u / \partial x_2^*$. At the optimum,

$$\frac{p_2}{p_1} = \frac{u_2}{u_1}$$

Use (1) and (2),

$$\frac{u_2}{u_1} = \frac{(x_1^\sigma + x_2^\sigma)^{\frac{1-\sigma}{\sigma}} x_2^{\sigma-1}}{(x_1^\sigma + x_2^\sigma)^{\frac{1-\sigma}{\sigma}} x_1^{\sigma-1}} = \left(\frac{x_2}{x_1} \right)^{\sigma-1}$$

Thus

$$\begin{aligned} \left(\frac{x_2^*}{x_1^*} \right)^{\sigma-1} &= \frac{p_2}{p_1} \\ \frac{x_1^*}{x_2^*} &= \left(\frac{p_2}{p_1} \right)^{-\frac{1}{\sigma-1}} \end{aligned} \quad (4)$$

We can use (4) to calculate,

$$\frac{d\left(\frac{x_1^*}{x_2^*}\right)}{d\left(\frac{p_2}{p_1}\right)} = -\frac{1}{\sigma-1} \left(\frac{p_2}{p_1} \right)^{\frac{\sigma}{1-\sigma}} \quad (5)$$

Then according to (3) we need

$$\frac{p_2/p_1}{x_1^*/x_2^*} = \frac{\frac{p_2}{p_1}}{\left(\frac{p_2}{p_1} \right)^{\frac{1}{\sigma-1}}} = \left(\frac{p_2}{p_1} \right)^{1-\frac{1}{\sigma-1}} = \left(\frac{p_2}{p_1} \right)^{\frac{\sigma-2}{\sigma-1}} \quad (6)$$

Multiply (5) and (6), following (3)

$$\frac{d\left(\frac{x_1^*}{x_2^*}\right)}{d\left(\frac{p_1}{p_2}\right)} \frac{p_1}{x_2^*} = \frac{1}{\sigma-1} \left(\frac{p_1}{p_2} \right)^{\frac{2-\sigma}{\sigma-1}} \left(\frac{p_1}{p_2} \right)^{\frac{\sigma-2}{\sigma-1}} = \frac{1}{\sigma-1}$$

We have verified it is CES function, the elasticity of substitution is $1/(\sigma-1)$

which is a constant.

1.1 General Case of CES

The most general functional form of CES is given by

$$F = A \left(\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

where A is a constant or a stochastic process (cf. technology parameter A in Cobb-Douglas production function), α denotes distribution parameter as the exponent in C-D function, besides we assume $\alpha_1 + \alpha_2 = 1$. In this case, σ is exactly the elasticity of substitution parameter as we have seen in first case as ε^{sub} . The purpose of the outer exponent $\frac{\sigma}{\sigma-1}$ is designed to render the function as first-order homogeneous (linear homogeneous). We can show this as follows:

$$\begin{aligned} F &= A \left(\alpha_1^{\frac{1}{\sigma}} (tx_1)^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} (tx_2)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \\ &= A \left\{ t^{\frac{\sigma-1}{\sigma}} \left[\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} \right] \right\}^{\frac{\sigma}{\sigma-1}} \\ &= tA \left(\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \end{aligned}$$

As the first case, we define the elasticity of substitution at optimum (point-elasticity) as

$$\varepsilon^{sub} = \frac{\frac{d(x_1^*/x_2^*)}{x_1^*/x_2^*}}{\frac{d(p_2/p_1)}{p_2/p_1}} = \frac{\frac{d(x_1^*/x_2^*)}{x_1^*/x_2^*}}{\frac{d(F_2/F_1)}{F_2/F_1}} = \frac{d\left(\frac{x_1^*}{x_2^*}\right)}{d\left(\frac{p_2}{p_1}\right)} \cdot \frac{p_2}{p_1} \cdot \frac{x_1^*}{x_2^*} \quad (7)$$

Take partial derivative w.r.t. x_1 and x_2 ,

$$\begin{aligned} F_1 &= A \frac{\sigma}{\sigma-1} \left[\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}-1} \frac{\sigma-1}{\sigma} \alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}-1} \\ &= A \alpha_1^{\frac{1}{\sigma}} x_1^{-\frac{1}{\sigma}} \left[\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma-1}} \\ &= A \alpha_1^{\frac{1}{\sigma}} x_1^{-\frac{1}{\sigma}} F^{\frac{1}{\sigma}} = A \alpha_1^{\frac{1}{\sigma}} \left(\frac{F}{x_1} \right)^{\frac{1}{\sigma}} \end{aligned} \quad (8)$$

Similarly,

$$F_2 = A \alpha_2^{\frac{1}{\sigma}} \left(\frac{F}{x_2} \right)^{\frac{1}{\sigma}} \quad (9)$$

Divide (9) by (8),

$$\frac{F_2}{F_1} = \left(\frac{\alpha_2}{\alpha_1} \right)^{\frac{1}{\sigma}} \left(\frac{x_1}{x_2} \right)^{\frac{1}{\sigma}} = \frac{p_2}{p_1} \quad (10)$$

Next we shall isolate $\frac{x_1}{x_2}$ on one side at point of optimum,

$$\begin{aligned}\frac{\alpha_2 x_1}{\alpha_1 x_2} &= \left(\frac{p_2}{p_1}\right)^\sigma \\ \frac{x_1^*}{x_2^*} &= \left(\frac{p_2}{p_1}\right)^\sigma \frac{\alpha_1}{\alpha_2}\end{aligned}\tag{11}$$

To get $\frac{p_2}{p_1} / \frac{x_1^*}{x_2^*}$, we modify (11),

$$\begin{aligned}\frac{x_1^*}{x_2^*} &= \frac{p_2}{p_1} \left(\frac{p_2}{p_1}\right)^{\sigma-1} \frac{\alpha_1}{\alpha_2} \\ \frac{\frac{x_1^*}{x_2^*}}{\frac{p_2}{p_1}} &= \left(\frac{p_2}{p_1}\right)^{\sigma-1} \frac{\alpha_1}{\alpha_2} \\ \frac{\frac{p_2}{p_1}}{\frac{x_1^*}{x_2^*}} &= \left(\frac{p_2}{p_1}\right)^{1-\sigma} \frac{\alpha_2}{\alpha_1}\end{aligned}\tag{12}$$

From (11), we get

$$\frac{d\left(\frac{x_1^*}{x_2^*}\right)}{d\left(\frac{p_2}{p_1}\right)} = \sigma \left(\frac{p_2}{p_1}\right)^{\sigma-1} \frac{\alpha_1}{\alpha_2}\tag{13}$$

According to (7), we multiply (13) and (12),

$$\sigma \left(\frac{p_2}{p_1}\right)^{\sigma-1} \frac{\alpha_1}{\alpha_2} \left(\frac{p_2}{p_1}\right)^{1-\sigma} \frac{\alpha_2}{\alpha_1} = \sigma$$

Thus we have confirmed the property of CES function. The larger the σ the greater the substitutability, if $\sigma = 0$, both products have to be in a fixed proportion, such as right and left shoes.

1.2 Cobb-Douglas Function As The Special Case

We will see in this section that Cobb-Douglas function is simply a limiting version of CES function, which renders C-D function as a special case. Let's go back to the general CES form,

$$F = A \left(\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}\tag{14}$$

when $\sigma \rightarrow 0$, (14) will approaches C-D function. However we cannot simply equal σ to 1 here, since the denominator of outer exponent is undefined. Thus, the most natural way to proceed is to use *L'Hôpital's* rule. First divide F by A and take natural log,

$$\ln \frac{F}{A} = \frac{\sigma \ln \left(\alpha_1^{\frac{1}{\sigma}} x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2^{\frac{1}{\sigma}} x_2^{\frac{\sigma-1}{\sigma}} \right)}{\sigma - 1} = \frac{m(\sigma)}{n(\sigma)} \quad (15)$$

To see whether it is $\frac{0}{0}$ form, we set $\sigma = 1$, the denominator obviously equals 0, and numerator can be shown

$$\ln (\alpha_1 x_1^0 + \alpha_2 x_2^0) = \ln 1 = 0$$

We can perform *L'Hôpital's* rule to (15), but we find it will become extremely messy in notation to handle this problem, since we have to use three times derivative product rule in a nested manner and exponent derivative rule. Actually it is unnecessary to work on such a general function to derive the C-D function, we can simplify (15) a bit and keep its CES features.

There will be some slight notation change in this simplified version,

$$F = A [\alpha K^{-\rho} + (1 - \alpha) L^{-\rho}]^{-\frac{1}{\rho}} \quad (16)$$

as you can see we replace x_1 and x_2 by capital K and labour L . A and α have their corresponding meaning in C-D function, but not ρ yet. Besides, we can notice that we have set $1 - \frac{1}{\sigma} = -\rho$ implicitly, if $\sigma = 1$ then $\rho = 1$. Thus if $\rho \rightarrow 0$, (16) approaches C-D function.

Divide (16) by A then take natural log,

$$\ln \frac{F}{A} = \frac{-\ln [\alpha K^{-\rho} + (1 - \alpha) L^{-\rho}]}{\rho} = \frac{m(\rho)}{n(\rho)}$$

To see the $\frac{0}{0}$, simply plug 0 into ρ numerator, $-\ln [\alpha + 1 - \alpha] = 0$. Next, use *L'Hôpital's* rule

$$\frac{m'(\rho)}{n'(\rho)} = \frac{-1}{\alpha K^{-\rho} + (1 - \alpha) L^{-\rho}} [-\alpha K^{-\rho} \ln K - (1 - \alpha) L^{-\rho} \ln L]$$

Then set $\rho = 0$, we get

$$\frac{\alpha \ln K + (1 - \alpha) \ln L}{\alpha + 1 - \alpha} = \alpha \ln K + (1 - \alpha) \ln L = \ln K^\alpha + \ln L^{1-\alpha} = \ln K^\alpha L^{1-\alpha}$$

Since we have divided by A and taking natural log, we need to perform the inverse operation to recover the CES form, thus

$$Ae^{\ln K^\alpha L^{1-\alpha}} = AK^\alpha L^{1-\alpha} \quad (17)$$

which is the standard form of Cobb-Douglas function.

1.3 CES Production Function

We have already seen the CES production function with input capital and labour can be written as:

$$F = A[\alpha K^{-\rho} + (1 - \alpha)L^{-\rho}]^{-\frac{1}{\rho}}$$

But question is what makes we find the CES production function should look like this? Simply put, why it looks like this? The origin of CES production function comes from Arrow et al (1961). In their paper, they tested two models

$$\begin{aligned} \frac{V}{L} &= c + dw + \eta \\ \ln \frac{V}{L} &= \ln a + b \ln w + \varepsilon \end{aligned}$$

where $V = F(K, L)$, w denotes the nominal wage rate per capita, L the labour input. They found that the log version of the model fit better with data across industries in U.S.. We turn $F(K, L)$ into per capita version: $y = f(k)$, where $\frac{K}{L} = k$ and $\frac{V}{L} = y$. Then MPK and MPL are $f'(k)$ and $\frac{f(k) - kf'(k)}{1}$ respectively. We will focus on the second model and derive the CES production function here. To start from the function without error

term

$$\ln y = \ln a + b \ln w \quad (18)$$

Substitute $w = f(k) - kf'(k)$ into last equation,

$$\ln y = \ln a + b \ln [f(k) - kf'(k)]$$

Solve for $f'(k)$,

$$\ln y - \ln a = b \ln [f(k) - kf'(k)]$$

$$\ln \frac{y}{a} = b \ln [y - kf'(k)]$$

$$\ln \left(\frac{y}{a} \right)^{\frac{1}{b}} = \ln [y - kf'(k)]$$

$$kf'(k) = y - \left(\frac{y}{a} \right)^{\frac{1}{b}}$$

$$kf'(k) = y - \frac{y^{\frac{1}{b}}}{a^{\frac{1}{b}}}$$

$$f'(k) = \frac{a^{\frac{1}{b}}y - y^{\frac{1}{b}}}{ka^{\frac{1}{b}}}$$

To make further modification in order to simplify the equation,

$$\frac{a^{\frac{1}{b}}y - y^{\frac{1}{b}}}{ka^{\frac{1}{b}}} = \frac{a^{-\frac{1}{b}}(a^{\frac{1}{b}}y - y^{\frac{1}{b}})}{k} = \frac{y - a^{-\frac{1}{b}}y^{\frac{1}{b}}}{k}$$

We define $\alpha = a^{-\frac{1}{b}}$, then

$$\frac{dy}{dk} = \frac{y - \alpha y^{\frac{1}{b}}}{k} = \frac{y(1 - \alpha y^{\frac{1}{b}-1})}{k} = \frac{y(1 - \alpha y^{\rho})}{k}$$

where we define $\rho = \frac{1}{b} - 1$. This is a separable differential equation, so separate y and k ,

$$\frac{dk}{k} = \frac{dy}{y(1 - \alpha y^{\rho})}$$

and perform partial fraction decomposition,

$$\frac{dk}{k} = \frac{dy}{y} + \frac{\alpha y^{\rho-1}dy}{1 - \alpha y^{\rho}}$$

Take integral on both sides,

$$\ln k = \ln y - \frac{1}{\rho} \ln(1 - \alpha y^\rho) + \frac{1}{\rho} \ln \beta$$

where $\ln \beta$ is constant term after indefinite integration. Take antilog on both sides then raise the power of ρ , we get

$$k^\rho = \frac{\beta y^\rho}{1 - \alpha y^\rho}$$

To solve for y ,

$$k^\rho - \alpha k^\rho y^\rho = \beta y^\rho$$

$$k^\rho = (\beta + \alpha k^\rho) y^\rho$$

$$y^\rho = \frac{k^\rho}{\beta + \alpha k^\rho}$$

$$y = k(\beta + \alpha k^\rho)^{-\frac{1}{\rho}}$$

$$y = [k^{-\rho}(\beta + \alpha k^\rho)]^{-\frac{1}{\rho}}$$

$$y = (\beta k^{-\rho} + \alpha)^{-\frac{1}{\rho}}$$

To recover the aggregate form,

$$F(K, L) = (\beta K^{-\rho} + \alpha L)^{-\frac{1}{\rho}}$$

2 Dixit-Stiglitz Lite

In this chapter we present the Dixit and Stiglitz (1977) monopolistic competition model, which is currently used in New-Keynesian DSGE model.

2.1 Frisch and Marshallian Demand Function

There are $n + 1$ goods, representative household maximises utility function

$$U \left[x_0, \left(\sum_{i=0}^n x_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \right]$$

by choosing a consumption plan $\mathbf{x} \in \mathbb{R}^{n+1}$. Where σ is the constant elasticity of substitution between monopolistic goods $i = 1, \dots, n$. x_0 is the numeraire good, which can be interpreted as labour or leisure time. With constraint,

$$x_0 + \sum_{i=1}^n p_i x_i = I$$

The representative consumer maximise utility function

$$U = \left(\int_0^n q(\omega)^\rho d\omega \right)^{\frac{1}{\rho}} \quad 0 < \rho < 1 \quad (19)$$

There is a continuum of goods, indexed by $\omega \in [0, n]$.¹ ω is the measure of substitutability. With budget constraint,

$$\int_0^n p(\omega) q(\omega) d\omega = I \quad (20)$$

where $p(\omega)$ is the price of good ω .

In order to maximise utility function easily, we take monotonic transformation to utility function

$$U^\rho = \left[\left(\int_0^n q(\omega)^\rho d\omega \right)^{\frac{1}{\rho}} \right]^\rho = \int_0^n q(\omega)^\rho d\omega$$

which would yield the same optimisation solution.

Form Lagrangian,

$$\begin{aligned} \mathcal{L}[q(\omega)] &= U^\rho - \lambda \left[\int_0^n p(\omega) q(\omega) d\omega - I \right] \\ &= \int_0^n q(\omega)^\rho d\omega - \lambda \left[\int_0^n p(\omega) q(\omega) d\omega - I \right] \end{aligned}$$

F.O.C.s are,

$$\frac{\partial \mathcal{L}}{\partial q(\omega)} = \rho q(\omega)^{\rho-1} - \lambda p(\omega) = 0$$

¹We model the numeraire good $q(0)$ implicitly which was also used in original Dixit-Stiglitz monopolistic competition model.

the integration sign is gone if we take a derivative w.r.t. a specific ω . Rearrange,

$$\begin{aligned}\rho q(\omega)^{\rho-1} &= \lambda p(\omega) \\ q(\omega)^{\rho-1} &= \frac{\lambda p(\omega)}{\rho} \\ q(\omega) &= \left(\frac{\lambda p(\omega)}{\rho} \right)^{\frac{1}{\rho-1}}\end{aligned}\tag{21}$$

(21) is ‘Frisch demand function’. Take ratio of Frisch demands for two different goods, ω_1 and ω_2 yields,

$$\begin{aligned}\frac{q(\omega_1)}{q(\omega_2)} &= \left(\frac{\lambda p(\omega_1)/\rho}{\lambda p(\omega_2)/\rho} \right)^{\frac{1}{\rho-1}} \\ &= \left(\frac{p(\omega_1)}{p(\omega_2)} \right)^{\frac{1}{\rho-1}}\end{aligned}\tag{22}$$

Now we define $\sigma = \frac{1}{1-\rho}$ to simplify notation. Multiply both sides by $p(\omega_1)q(\omega_2)$,

$$p(\omega_1)q(\omega_1) = q(\omega_2)p(\omega_1)^{1-\sigma}p(\omega_2)^\sigma$$

Integrate both sides out of ω_1 ,

$$\int_0^n p(\omega_1)q(\omega_1) d\omega_1 = \int_0^n q(\omega_2)p(\omega_1)^{1-\sigma}p(\omega_2)^\sigma d\omega_1$$

The left-hand side is I , then

$$I = \int_0^n q(\omega_2)p(\omega_1)^{1-\sigma}p(\omega_2)^\sigma d\omega_1$$

To isolate $q(\omega_2)$,

$$I = q(\omega_2) \int_0^n p(\omega_1)^{1-\sigma}p(\omega_2)^\sigma d\omega_1$$

Then we get Marshallian demand function:

$$q(\omega_2) = \frac{I}{\int_0^n p(\omega_1)^{1-\sigma}p(\omega_2)^\sigma d\omega_1} = \frac{Ip(\omega_2)^{-\sigma}}{\int_0^n p(\omega_1)^{1-\sigma} d\omega_1}\tag{23}$$

2.2 Indirect Utility Function and CPI

Plug (23) into (19) to get optimised indirect utility function,

$$\begin{aligned} U^* &= \left\{ \int_0^n \left[\frac{Ip(\omega_2)^{-\sigma}}{\int_0^n p(\omega_1)^{1-\sigma} d\omega_1} \right]^\rho d\omega \right\}^{\frac{1}{\rho}} \\ &= \left[\frac{I^\rho}{\left(\int_0^n p(\omega_1)^{1-\sigma} d\omega_1 \right)^\rho} \int_0^n p(\omega_2)^{\frac{\rho}{\rho-1}} d\omega_2 \right]^{\frac{1}{\rho}} \end{aligned}$$

Because $1 - \sigma = \frac{\rho}{\rho-1}$, thus

$$U^* = \left[\frac{I^\rho}{\left(\int_0^n p(\omega_1)^{1-\sigma} d\omega_1 \right)^\rho} \int_0^n p(\omega_2)^{1-\sigma} d\omega_2 \right]^{\frac{1}{\rho}}$$

Set utility equal to one and solve for I ,

$$\begin{aligned} 1 &= \left[\frac{I^\rho}{\left(\int_0^n p(\omega_1)^{1-\sigma} d\omega_1 \right)^\rho} \int_0^n p(\omega_2)^{1-\sigma} d\omega_2 \right]^{\frac{1}{\rho}} \\ 1 &= \frac{I}{\int_0^n p(\omega_1)^{1-\sigma} d\omega_1} \left(\int_0^n p(\omega_2)^{1-\sigma} d\omega_2 \right)^{\frac{1}{\rho}} \end{aligned}$$

and we can replace $\rho = \frac{\sigma-1}{\sigma}$,

$$1 = I \cdot \frac{\left(\int_0^n p(\omega_2)^{1-\sigma} d\omega_2 \right)^{\frac{\sigma}{\sigma-1}}}{\int_0^n p(\omega_1)^{1-\sigma} d\omega_1}$$

switch ω_2 to ω_1 ,

$$I = \frac{\int_0^n p(\omega_1)^{1-\sigma} d\omega_1}{\left(\int_0^n p(\omega_1)^{1-\sigma} d\omega_1 \right)^{\frac{\sigma}{\sigma-1}}} = P \quad (24)$$

$$P = \left(\int_0^n p(\omega_1)^{1-\sigma} d\omega_1 \right)^{\frac{1}{1-\sigma}} \quad (25)$$

where P denotes the expenditure to purchase a unit-level utility. (25) is also the consumer price index (CPI) in this model.

2.3 Consumption Diversity

We assume all varieties have the same price and quantity, thus

$$\int_0^n p(\omega)q(\omega) d\omega = npq = I$$

We can see easily,

$$U = \left(\int_0^n q(\omega)^\rho d\omega \right)^{\frac{1}{\rho}} = (nq^\rho)^{\frac{1}{\rho}} = \left[n \left(\frac{I}{np} \right)^\rho \right]^{\frac{1}{\rho}} = n^{\frac{1}{\sigma-1}} \frac{I}{P}$$

as you can see $\frac{\partial U}{\partial n} > 0$, utility increases as n goes up.

2.4 Optimum Pricing

If you have some experiences of New-Keynesian (NK) DSGE models, you will recall that NKs usually employ Calvo pricing mechanism to derive the optimum price and New-Keynesian Phillips curve (NKPC). Because the model assume monopolistic competition, every firm and consumer has some monopoly power on its price and wage. And we see the most primitive version of optimum pricing here under Dixit-Stiglitz formulation.

This is a static model, no time evolution, we define a firm's profits as

$$\pi = pq - w(cq + f) = pq - l(q) \quad (26)$$

where c is constant marginal cost, w denotes real wage rate, $l(q)$ is the labour demand function with argument q . The firm will choose a p to maximise its profits, set up F.O.C.,

$$\frac{\partial \pi}{\partial p} = q + p \frac{\partial q}{\partial p} - wc \frac{\partial q}{\partial p}$$

the first two terms are from derivative product rule. Solve for p ,

$$p = wc - \frac{q}{\frac{\partial q}{\partial p}} \quad (27)$$

Use Marshallian demand function (23) and CPI (25) we can get

$$q(\omega) = p(\omega)^{-\sigma} P^{\sigma-1} I \quad (28)$$

take partial derivative,

$$\frac{\partial q(\omega)}{\partial p(\omega)} = -\sigma p(\omega)^{-\sigma-1} P^{\sigma-1} I \quad (29)$$

Divide (28) by (29) we can get the second term of (27),

$$\frac{-p(\omega)^{-\sigma} P^{\sigma-1} I}{-\sigma p(\omega)^{-\sigma-1} P^{\sigma-1} I} = \frac{p}{\sigma}$$

Substitute back to (27),

$$p = wc + \frac{p}{\sigma}$$

Again, solve for p ,

$$\begin{aligned} p \left(1 + \frac{1}{\sigma} \right) &= wc \\ p \left(\frac{\sigma + 1}{\sigma} \right) &= wc \end{aligned}$$

It is clear that the price should be a proportional mark-up over cost wc .

And mark-up level depends on substitutability σ , if $\sigma \rightarrow \infty$ then $p \rightarrow wc$.

End.

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