Ordinary Differential Equations

Weijie Chen

Department of Political and Economic Studies

University of Helsinki

16 Aug, 2011

Contents

Abstract

Differential equations are tools describing dynamic systems, which are changing as time flows. Learning differential equation will finally bring you into advanced study of any science. Our calculus knowledge is completed by learning differential equation. So it won't be exagerating to state that differential equation is the most important fundamental mathematical tool you need to step into advanced study. We have several other tools to study dynamic systems, such as difference equations, calculus of variations, optimal control theory, dynamic programming, and etc. But differential equations is the start of all this, their central ideas is built on the basic idea that how differential equations describe the dynamic world.

1 First Order Differential Equations

Most important classes of differential equations with explicit solution[1] will be discussed here. We will study the solutions first, then look at different useful types of differential equation.

1.1 Constant Coeffient Case

The standard form of first order differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t), \qquad \text{or} \qquad y' + p(t)y = q(t)$$

where $p(t)$ and $q(t)$ are continuous functions. But in this section, we only tackle the constant coefficient case. If $q(t) = 0$, this differential equation is called *homogeneous*[2], $q(t) \neq 0$ *nonhomogeneous*. We will look at them seperately.

[1] Some differential equations do have closed-form solution, but it is not our concern here.

[2] The name is from linear algebra, because if you multiply a constant on both sides, it remains valid.

1.1.1 The Homogeneous

Say we have a homogeneous first-order linear differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}t} + ay = 0 \tag{1}$$

You need nothing more here, only your basic integral calculus knowledge,

$$\begin{aligned} \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} &= -a \\ \int \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t &= \int (-a)\,\mathrm{d}t \\ \int \frac{1}{y}\,\mathrm{d}y &= -at + C_1 \\ \ln|y| + C_2 &= -at + C_1 \end{aligned}$$

assuming $y > 0$,

$$\begin{aligned} y &= e^{-at+C_1-C_2} \\ y &= e^{-at}e^{C_1-C_2} \\ y &= Ae^{-at} \end{aligned}$$

where $A = e^{C_1-C_2}$ is a constant. This is the start of everything, everything in differential equation is built on solution $y = Ae^{-at}$, later on we will see how central it play the role in our solution method.

We change the notation slightly $y = Ae^{rt}$, we know that all solution will have this form, the first derivative is $y' = rAe^{rt}$, put them back to the equation (1),

$$rAe^{rt} + aAe^{rt} = 0$$

factor Ae^{rt},

$$(r + a)Ae^{rt} = 0$$

We don't Ae^{rt} is positive or not, but we know $Ae^{rt} \neq 0$ for sure, because $A \neq 0$,

$$r + a = 0$$
$$r = -a$$

Then our solution is Ae^{-at}, we are right back to where we started, but it is a good start. We will frequently make use of Ae^{rt} to represent the *general solution* of homogeneous first-order linear differential equation. If we can use *initial condition* figure out A, we have a *definite solution*.[3]

1.1.2 The Nonhomogeneous

Here is a nonhomogeneous one,

$$\frac{\mathrm{d}y}{\mathrm{d}t} + ay = b.$$

The general solution is $y = y_c + y_p$, where y_c is *complementary solution* of its *complementary equation*, which is simply its homogeneous version,

$$\frac{\mathrm{d}y}{\mathrm{d}t} + ay = 0,$$

and y_p is a particular solution of the nonhomogeneous version. I know you've got question, 'why would we have such a strange look general solution?' The simple proof is coming soon, but here we should understand what is particular solution and how it is conceptually different from definite solution.

A particular solution undoubtedly will satisfy the differential equation, even if a constant. Thus, then why don't we try a constant, say $y_p = k$, then $y_p' = 0$, the nonhomogeneous differential equation becomes,

$$ay_p = b$$
$$y_p = \frac{b}{a}.$$

[3] It is not *particular solution*, it will be clear soon.

This is our particular solution, because it is particular so it is 'a' not 'the', particular solution. According to the formula of general solution of nonhomogeneous case, we have

$$y = Ae^{-at} + \frac{b}{a}$$

Say we have an intial condition $y(0) = 1$,

$$\begin{aligned} Ae^{-a0} + \frac{b}{a} &= 1 \\ A &= 1 - \frac{b}{a} \end{aligned}$$

Plug it into general solution, we have a definite solution,

$$y = \left(1 - \frac{b}{a}\right)e^{-at} + \frac{b}{a}$$

I guess you have known the conceptual difference between definite solution and particular solution after study this example. Just remember definite solution is found by using initial condition.

Now we can see a proof of general solution. We are give y_p and y, we have two equations,

$$\begin{aligned} y' + ay &= b \\ y_p' + ay_p' &= b \end{aligned}$$

Substracting,

$$\begin{aligned} y' + ay - y_p' - ay_p' &= 0 \\ (y - y_p)' + a(y - y_p) &= 0 \end{aligned}$$

So $y - y_p$ is the solution of its homogeneous version, it has the form,

$$\begin{aligned} y - y_p &= Ae^{rt} \\ y =& Ae^{rt} + y_p \end{aligned}$$

where $Ae^{rt} = y_c$, so it is proved.

1.1.3 An Example

Find the definite solution, where $y(0) = 9$

$$\frac{\mathrm{d}y}{\mathrm{d}t} - 2y = 5.$$

First particular solution, we choose the simplest $y = k$ case,

$$-2k = 5$$
$$y_p = k = -\frac{5}{2}$$

Next complementary solution, use $y = Ae^{rt}$

$$\frac{\mathrm{d}y}{\mathrm{d}t} - 2y = 0$$
$$rAe^{rt} - 2Ae^{rt} = 0$$
$$(r-2)Ae^{rt} = 0$$
$$r = 2$$

So the complementary solution is

$$y_c = Ae^{2t}$$

Acorrding to formula $y = y_c + y_p$, the general solution is

$$y = Ae^{2t} - \frac{5}{2}$$

But this isn't over yet, we have initial condition,

$$A - \frac{5}{2} = 9$$
$$A = \frac{23}{2}$$

The definite solution is

$$y = \frac{23}{2}e^{2t} - \frac{5}{2}$$

1.2 Variable Coefficient Case

More general case of first-order differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}t} - p(t)y = q(t)$$

We have studie the constant coefficient case, here we will study variable coefficient case. But first we need to study *exact differential equation*, our intention will be clear soon.

1.2.1 Exact Differential Equations

Recall the total differential of a bivariate function we have studied in calculus course,

$$\mathrm{d}F(x, y) = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y$$

If we set $\mathrm{d}F = 0$,

$$\frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y = 0$$

This is an exact differential equation. Conventionlly we denote $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$,

$$M\mathrm{d}x + N\mathrm{d}y = 0$$

By *Clairaut's theorem*[4],

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Because $\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial x \partial y}$.

For instance, we have

$$3x^2y\,\mathrm{d}x + (x^3 + 2y)\,\mathrm{d}y = 0$$

$M = 3x^2y$ and $N = x^3+2y$, we can verify it is an exact differential equation,

$$\frac{\partial M}{\partial y} = 3x^2$$
$$\frac{\partial N}{\partial x} = 3x^2$$

[4] It is a generalization of *Young's theorem*, I prefer to using general case.

For any exact differential equations $\mathrm{d}F(x, y) = 0$, its solution is $F(x, y) = c$. So to solve an exact differential equation is to find the $F(x, y)$, the solution method will be discussed here.

If we only integrate x out of M, but only x, we will have

$$F(x, y) = \int M \,\mathrm{d}x + \phi(y) \tag{2}$$

where $\phi(y)$ is a function of y, we don't know it definite form yet. We add it here, because the partial derivative of F with respect to x will take y as constant, we simply reinstate it here. We differentiate this equation on both sides with y,

$$\frac{\partial F}{\partial y} = \int \frac{\partial M}{\partial y} \,\mathrm{d}x + \phi'(y) \tag{3}$$

Because $\frac{\partial F}{\partial y} = N$, then

$$N = \int \frac{\partial M}{\partial y} \,\mathrm{d}x + \phi'(y)$$

$$\phi'(y) = N - \int \frac{\partial M}{\partial y} \,\mathrm{d}x \tag{4}$$

Last step is to integrate $\phi(y)$ with respect to y,

$$\phi(y) = \int N \,\mathrm{d}y - \int\int \frac{\partial M}{\partial y} \,\mathrm{d}x \,\mathrm{d}y \tag{5}$$

I know, this looks not obvious at all, so better look at an example.

Solve $3x^2 y \,\mathrm{d}x + (x^3 + 2y) \,\mathrm{d}y = 0$. Integrate out y of M, apply equation (2),

$$\begin{aligned} F(x, y) &= \int 3x^2 y \,\mathrm{d}x + \phi(y) \\ &= x^3 y + \phi(y) \end{aligned}$$

We don't need to worry about constant, it is omitted, or you can say it is merged into $\phi(y)$. Follow equation (3), we take derivative with respect to y,

$$\frac{\partial F}{\partial y} = x^3 + \phi'(y)$$

Go on, use equation (4), and because $\frac{\partial F}{\partial y} = N$

$$\begin{aligned} \phi'(y) &= x^3 + 2y - x^3 \\ &= 2y \end{aligned}$$

Then integrate,

$$\int \phi'(y)\, \mathrm{d}y = y^2$$

Again we omit the constant, $\phi(y) = y^2$, so we have $F(x, y)$,

$$F(x, y) = x^3 y + y^2$$

which implies the solution of given differential equation is

$$x^3 y + y^2 = c$$

Verify by taking total derivative,

$$\frac{\partial}{\partial x}(x^3 y + y^2)\, \mathrm{d}x + \frac{\partial}{\partial y}(x^3 y + y^2)\, \mathrm{d}y = 3x^2 y\, \mathrm{d}x + (x^3 + 2y)\, \mathrm{d}y$$

Question solved. Spend some time chewing over it, since it is not intuitive.

1.2.2 Integrating Factor

It turns out that the procedures above can deal with differential equation even when they are not exact. The function of *integrating factor* is to make inexact differential equation exact. To understand the mechanism, say we have a variable coefficient differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + px = q$$

where p and q are functions of y, and which can be written as

$$\mathrm{d}y + (px - q)\mathrm{d}x = 0. \tag{6}$$

Obviously, this is not exact. We intend to multiply both sides by an integrating factor I, to make it exact.

$$\overbrace{I}^{M}\,\mathrm{d}y + \overbrace{I(px-q)}^{N}\,\mathrm{d}x = 0. \tag{7}$$

Because $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then

$$\frac{\partial N}{\partial x} = Ip,$$

equate it with $\frac{\partial M}{\partial y}$,

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\mathrm{d}I}{\mathrm{d}y} = Ip\\ \frac{\mathrm{d}I/\mathrm{d}y}{I} &= \frac{I'}{I} = p\end{aligned}$$

We can solve it by seperating variable, which will be studied later, but the idea is very easy,

$$\begin{aligned}\frac{\mathrm{d}I/\mathrm{d}y}{I} &= p\\ \frac{\mathrm{d}I}{I} &= p\,\mathrm{d}y\\ \int\frac{\mathrm{d}I}{I} &= \int p\,\mathrm{d}y\\ \ln I &= \int p\,\mathrm{d}y\\ I &= e^{\int p\,\mathrm{d}y}\end{aligned}$$

This is what we called integrating factor, it is far from intuitive judging from its appearance. Next step we substitute it back to (7),

$$e^{\int p\,\mathrm{d}y}\mathrm{d}y + e^{\int p\,\mathrm{d}y}(px-q)\mathrm{d}x = 0$$

solve it, by method of solving exact differential equation,

$$\begin{aligned}F(x,y) &= \int e^{\int p\,\mathrm{d}y}\mathrm{d}x + \phi(y) = xe^{\int p\,\mathrm{d}y} + \phi(y)\\ \frac{\partial F}{\partial y} &= xe^{\int p\,\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}y}\int p\mathrm{d}y + \phi'(y)\\ \frac{\partial F}{\partial y} &= xpe^{\int p\,\mathrm{d}y} + \phi'(y)\end{aligned}$$

$\frac{\partial F}{\partial y} = N$, so

$$
\begin{aligned}
xpe^{\int p \, \mathrm{d}y} + \phi'(y) &= e^{\int p \, \mathrm{d}y}(px - q) \\
\phi'(y) &= xpe^{\int p \, \mathrm{d}y} - e^{\int p \, \mathrm{d}y} q - xpe^{\int p \, \mathrm{d}y} \\
\phi'(y) &= -qe^{\int p \, \mathrm{d}y} \\
\phi(y) &= -\int qe^{\int p \, \mathrm{d}y} \, \mathrm{d}y
\end{aligned}
$$

We don't know the definite form of p or q so, nothing can be done further. Back to $F(x, y)$,

$$
\begin{aligned}
F(x, y) &= xe^{\int p \, \mathrm{d}y} + \phi(y) \\
F(x, y) &= xe^{\int p \, \mathrm{d}y} - \int qe^{\int p \, \mathrm{d}y} \, \mathrm{d}y
\end{aligned}
$$

which implies the solution is

$$
xe^{\int p \, \mathrm{d}y} - \int qe^{\int p \, \mathrm{d}y} \, \mathrm{d}y = c
$$

Rearrange,

$$
\begin{aligned}
xe^{\int p \, \mathrm{d}y} &= c + \int qe^{\int p \, \mathrm{d}y} \, \mathrm{d}y \\
x &= e^{-\int p \, \mathrm{d}y} \left(c + \int qe^{\int p \, \mathrm{d}y} \, \mathrm{d}y \right)
\end{aligned}
$$

If the original inexact differential equation is written as

$$
\frac{\mathrm{d}x}{\mathrm{d}y} + py = q
$$

Then

$$
y = e^{-\int p \, \mathrm{d}x} \left(A + \int qe^{\int p \, \mathrm{d}x} \, \mathrm{d}x \right)
$$

where we simply change the symbol c into A which is arbitraty constant.

Solving exact differential equations is a technique, not much of insight or intuition, although it won't be the high place of your learning list topics about differential equation, it indeed practice your skill of calculus. So that is why I said in the beginning, learning differential equation will complete your knowledge of calculus.

1.2.3 The Homogeneous

After study exact differential equations and integrating factor, rest of content in this section is just byproduct of previous knowledge, you will see soon. We have a differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = 0$$

Solve it as we have seen before,

$$\begin{aligned}
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} &= -p(t) \\
\int \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t &= -\int p(t)\,\mathrm{d}t \\
\int \frac{1}{y}\mathrm{d}y &= -\int p(x)\,\mathrm{d}x \\
\ln y &= -c - \int p(t)\,\mathrm{d}t \\
y &= e^{-c-\int p(t)\,\mathrm{d}t} \\
y &= Ae^{-\int p(t)\,\mathrm{d}t},
\end{aligned}$$

where $A^{-c} = e^{-c}$.

Well, if this is not obvious to you, let me reproduce the result from constant coeffient differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}t} + ay = 0 \qquad y = Ae^{-at}$$

Actually,

$$y = Ae^{-at} = Ae^{-\int a\mathrm{d}t}$$

Now I guess it can't be more obvious than this, when you compare with $y = Ae^{-\int p(t)\,\mathrm{d}t}$.

1.2.4 The Nonhomogeneous

When we are discussing about integrating factor, we use an example,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + px = q.$$

The basic idea it to turn any inexact differential equation into exact one by multiplying an integrating factor, which of course is $e^{\int p \mathrm{d}y}$. Then solve it by method of exact equations. You will finally get,

$$y = e^{-\int p \, \mathrm{d}x} \left(A + \int q e^{\int p \, \mathrm{d}x} \, \mathrm{d}x \right)$$

This is exactly we had seen (1.2.2). Whether you want to use this result or not, it is better understand how we arrive at this conclusion.

1.3 Seperable Equation

We have used a little trick of serpable equation before, it is very easy to understand how it works, we just seperate different variables on different side and take integral. We will see an example to explain it. There is a famous population growth model set under the assumption of *Malthus' Law*,

$$y'(t) = ky(t).$$

This model is considerably simple, which means the growth rate of population is proportional to population size and dependent of time (we call this *time-dependent differential equation*). The steps below are how we solve a seperable differential equation generally, soon you will understand what we mean 'seperable'.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= ky \\ \mathrm{d}y &= ky\mathrm{d}t \\ \frac{\mathrm{d}y}{y} &= k\mathrm{d}t \end{aligned}$$

Here, we have different variables on different sides, then integrate,

$$\begin{aligned} \int \frac{1}{y} \, \mathrm{d}y &= k \int \, \mathrm{d}t \\ \ln |y| &= kt + C \\ |y| &= e^{kt+C} \end{aligned}$$

Because $e^{kt+C} > 0$

$$y = e^{kt}e^{C}$$

Set the constant $A = e^{C}$

$$y(t) = Ae^{kt}$$

In words, if the environment has unlimited resources and space, population[5] will multiply in an exponential speed. Generally it is called *law of natural growth*, Malthus' law is an applied example of it, if $A < 0$ it is *law of natural decay*, it is used in describing such as, half-life. To see the significance of A,

$$y(0) = Ae^{0} = A$$

A is the value at time 0, you can take it as initial value.

However it is not always realistic, so there is a model, named *logistic model*[6], has a more realistic description of evolution of growth path. We will look at an example next.

1.3.1 Logistic Differential Equation

As we have mentioned above, to assume a unlimited environment is not realistic, then what would happen if we set up a model under conditions limit resource and space. One widely-accepted result is the population will level off when reaching *carrying capacity* of environment. The simplest version of such model is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(1 - \frac{P}{K}).$$

This is logistic differential equation, P is population and K carrying capacity. It is easy to interprete the equation, if $P \to K$, population reaches

[5] I do not refer merely to human beings exclusively, it also applies to amimals, viruses, microbes.

[6] It is extensively used in econometrics in binary dependent varible model theory.

carrying capacity, population will not have enough resource and space to multiply at the initial speed, so $\frac{\mathrm{d}P}{\mathrm{d}t} \to 0$. If $P > K$, it becomes a decay function. The reason I brought up logistic equation, because it is easy to solve by method of seperation.

$$\int \frac{1}{P(1-P/K)}\mathrm{d}P = \int k\mathrm{d}t$$

We need *partial fractions* to evalutate the right side of the equation, first rewrite

$$\frac{1}{P(1-\frac{P}{K})} = \frac{1}{P(K-P)\frac{1}{K}} = \frac{K}{P(K-P)}.$$

Then,

$$\frac{K}{P(K-P)} = \frac{A}{P} + \frac{B}{K-P},$$

multiply both sides by $P(K-P)$,

$$K = A(K-P) + BP = AK - AP + BP = AK - (A-B)P,$$

it is easy to see that $A = 1$ and $B = 1$,

$$\begin{aligned}
\int \left(\frac{1}{P} + \frac{1}{K-P}\right)\mathrm{d}P &= k\int \mathrm{d}t \\
\int \frac{1}{P}\mathrm{d}P - \int \frac{1}{P-K}\mathrm{d}P &= k\int \mathrm{d}t \\
\ln|P| - \ln|K-P| &= kt + C \\
-(\ln|K-P| - \ln|P|) &= kt + C \\
\ln\left|\frac{K-P}{P}\right| &= -kt - C \\
\left|\frac{K-P}{P}\right| &= e^{-kt-C} \\
\frac{K-P}{P} &= Ae^{-kt}
\end{aligned}$$

where $A = e^{-C}$. We can find value of A, by letting $t = 0$,

$$\frac{A - P_0}{P_0} = A$$

Continue, we solve for P,

$$\frac{K}{P} - 1 = Ae^{-kt}$$
$$\frac{P}{K} = \frac{1}{1 + Ae^{-kt}}$$
$$P = \frac{K}{1 + Ae^{-kt}}$$

2 Second Order Differential Equations

Standard form of 2^{nd} order ODE is

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x).$$

where $p(x)$, $q(x)$ and $g(x)$ are continuous functions, this is nonhomogeneous ODE. If we set $g(x) = 0$,

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0,$$

and this is called second-order homogeneous differential equation.

2.1 Constant Coeffient Case-Nonhomogeneous

However most of time, we assume that the coefficients are constants, then most frequent form of second-order differential equation is written as

$$y''(t) + a_1 y'(t) + a_2 y = b \tag{8}$$

where a_1, a_2 and b are constants, if $b = 0$ it becomes a second-order homogeneous differential equation. There is no need to give a seperate section illustrating how to solve second-order homogeneous differential equation, because if you want to solve a second-order nonhomogeneous differential equation, you must solve its homogeneous version (complementary equation) during the solving process. For instance, we intend to solve equation (8), its comlemenatry equation is

$$y''(t) + a_1 y'(t) + a_2 y = 0 \tag{9}$$

If we denote the general solution of (8) $y(t)$,

$$y(t) = y_c + y_p$$

where y_c is the solution of complementary equation (9) and y_p is the particular solution of (8).

2.1.1 Complementary Solution-Homogeneous

From the previous knowledge of first-order differential equation, the solution of homogeneous one has the form $y(t) = Ae^{rt}$. Then we will try this solution in second-order equation, first compute,

$$y'(t) = rAe^{rt} \qquad y''(t) = r^2Ae^{rt}$$

Then complementary equation becomes,

$$\begin{aligned} r^2Ae^{rt} + a_1rAe^{rt} + a_2Ae^{rt} &= 0 \\ (r^2 + a_1r + a_2)Ae^{rt} &= 0 \end{aligned}$$

For simplicity, we usually assume that $Ae^{rt} \neq 0$, then

$$r^2 + a_1r + a_2 = 0$$

This is *characteristic equation*, and solve for r by the root formula,

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

The problem comes, if $a_1^2 - 4a_2 > 0$, we have two distince r, which means, there will be $y_1 = A_1e^{r_1t}$ and $y_2 = A_2e^{r_2t}$, two solutions. The general solution will either be randomly pick up one, or use a linear combination. The first proposal does not work, we need two constants, A_1 and A_2, because during the process of two times differentiation, we lost two constants, we should take this chance to reinstate both of them here. So the complementary solution should be a linear combination of two linearly independent solution, $y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$. Besides, we will encounter $a_1^2 - 4a_2 = 0$ or $a_1^2 - 4a_2 < 0$, we will discuss them seperately below.

Case I $a_1^2 - 4a_2 > 0$ We have two distinct roots, $r_1 \neq r_2$, we can simply write the complementary solution as,

$$y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

then $e^{r_1 t}$ and $e^{r_2 t}$ are linear independent.

Case II $a_1^2 - 4a_2 = 0$ When this is the case, we have

$$r = r_1 = r_2 = -\frac{a_1}{2}$$

Then look at y_c,

$$y_c = A_1 e^{rt} + A_2 e^{rt} = A_3 e^{rt}$$

where $A_1 + A_2 = A_3$. But one constant is gone, we are supposed to reinstate another one, say A_4. Here is the same trick as we did in first-order differential equation, recall that when using k is not probable, we use kt, then the same here we switch to $A_4 t e^{rt}$. The general solution is

$$y_c = A_3 e^{rt} + A_4 t e^{rt}$$

Case III $a_1^2 - 4a_2 < 0$ Obviously we will have a solution of complex numbers. Take a closer look at root formula, it can be rewritten as,

$$r = -\frac{a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_2}}{2} = -\frac{a_1}{2} \pm i \frac{\sqrt{4a_2 - a_1^2}}{2} = \alpha \pm i\beta$$

where $\alpha = -\frac{a_1}{2}$ and $\beta = \frac{\sqrt{4a_2 - a_1^2}}{2}$. Use genenal solution,

$$y_c = A_1 e^{(\alpha + i\beta)t} + A_2 e^{(\alpha - i\beta)t} = A_1 e^{\alpha t} e^{i\beta t} + A_2 e^{\alpha t} e^{-i\beta t}$$

Recall the famous *Euler's formula*[7].

$$e^{i\theta} = \cos\theta + i\sin\theta$$

[7] Refer to my notes *Complex Numbers*[1], where detailed derivation is given.

Then

$$
\begin{aligned}
y_c &= A_1 e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)] + A_2 e^{\alpha t}[\cos(\beta t) - i\sin(\beta t)] \\
&= A_1 e^{\alpha t}\cos(\beta t) + A_1 e^{\alpha t} i\sin(\beta t) + A_2 e^{\alpha t}\cos(\beta t) - A_2 e^{\alpha t} i\sin(\beta t) \\
&= e^{\alpha t}[A_1\cos(\beta t) + A_1 i\sin(\beta t) + A_2\cos(\beta t) - A_2 i\sin(\beta t)] \\
&= e^{\alpha t}[(A_1 + A_2)\cos(\beta t) + (A_1 - A_2) i\sin(\beta t)]
\end{aligned}
$$

or you can write,

$$= e^{\alpha t}[C_1\cos(\beta t) + C_2 i\sin(\beta t)]$$

where $C_1 = A_1 + A_2$ and $C_2 = A_1 - A_2$.

All above are talking about complementary solution, however actually we are also talking about homogeneous case, as I said before really not necessary to write a seperate section for it.

2.1.2 Particular Solution

Say we have

$$y''(t) - a_1 y'(t) + a_2 y(t) = b.$$

As we did in first-order case, any solution y if it works will do. The simplest case is $y_p = k$, then $y' = 0$ and $y'' = 0$,

$$a_2 k = b \qquad y_p = k = \frac{b}{a_2}$$

But what if $a_2 = 0$, then you should use $y_p = kt$, thus, $y' = k$ and $y'' = 0$,

$$a_1 k = b \qquad k = \frac{b}{a_1}$$

Then in this case $y_p = \frac{b}{a_1}t$. This isn't over, what if $a_1 = 0$ too? Then you raise the power of t, $y_p = kt^2$ this time. Then $y' = 2tk$ and $y'' = 2k$,

$$2k = b \qquad k = \frac{b}{2}$$

put them back, we get $y_p = \frac{b}{2}t^2$.

2.1.3 Time Path

Here are the solution of three cases reproduced,

$$
\begin{aligned}
a_1^2 - 4a_2 > 0 \qquad & y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t} \\
a_1^2 - 4a_2 = 0 \qquad & y_c = A_3 e^{rt} + A_4 t e^{rt} \\
a_1^2 - 4a_2 < 0 \qquad & y_c = e^{\alpha t}[C_1 \cos(\beta t) + C_2 \sin(\beta t)]
\end{aligned}
$$

And according general solution is

$$y_p + y_c = y_p + A_1 e^{r_1 t} + A_2 e^{r_2 t} \tag{10}$$

$$y_p + y_c = y_p + A_3 e^{rt} + A_4 t e^{rt} \tag{11}$$

$$y_p + y_c = y_p + e^{\alpha t}[C_1 \cos(\beta t) + C_2 \sin(\beta t)] \tag{12}$$

Look at (10), ask yourself what would the result be if $r_1 > 0$ and $r_2 > 0$ and $t \to \infty$. The solution does not converge, it will explode. In formal words, it has no *stable intertemporal equilibrium*. The only way or necessary-and-sufficient condition to ensure its convergence is $r_1 < 0$ and $r_2 < 0$. And you will even find (11) also must have this condition $r < 0$ to guarantee its convergence.

Now let's look the (12), how can we make it converge if $t \to \infty$? The only way is suppress α below zero, and what is α? It is the real part of the solution. So with these knowledge, you can conclude that no matter what kind of solution you have, the necessary-and-sufficient condition of convergence is the real part of the solution is negative [8].

2.2 Variable Term Case-Nonhomogenous

Note that the title of this section might be little confusing, actually we are still dealing with constant coefficient, but with a variable term. Here is the

[8] Because real number is a subset of complex number, real numbers just have them imaginary part equal zero.

general form,

$$y''(t) + a_1 y'(t) + a_2 y = g(t)$$

The term on the right-hand side of the equation is not a constant term any more, which we always assume to be constant till now. Here $g(t)$ is a continuous function. Although the complementary solution will not change, because we are always solving its homogeneous version which has nothing to do with variable term, we have to invent some new ways to find their particular solutions. We will discuss two methods here.

2.2.1 Method of Undetermined Coefficients

Better use an example to start. Solve

$$y''(t) + y'(t) - 2y(t) = t^3 \tag{13}$$

First, solve characteristic equation,

$$r^2 + r - 2 = 0$$
$$(r-1)(r+2) = 0$$

with roots $r_1 = 1$ and $r_2 = -2$. So the complementary solution is

$$y_c = A_1 e^t + A_2 e^{-2t}$$

Now, particular solution. Since $g(t) = t^3$, we seek a particular solution of form $y_p = c_1 t^3 + c_2 t^2 + c_3 t + c_4$, where c_i is undertermined coefficient. So $y_p' = 3c_1 t^2 + 2c_2 t + c_3$ and $y_p'' = 6c_1 t + 2c_2$. Plug them back to differential equation,

$$6c_1 t + 2c_2 + 3c_1 t^2 + 2c_2 t + c_3 - 2c_1 t^3 - 2c_2 t^2 - 2c_3 t - 2c_4 = t^3$$

Rearrange the power terms in descending order,

$$-2c_1 t^3 + (3c_1 - 2c_2)t^2 + (6c_1 + 2c_2 - 2c_3)t^{+} 2c_2 + c_3 - 2c_4 = t^3,$$

we can set up a linear equation system to determine all coefficients,

$$-2c_1 = 1$$

$$3c_1 - 2c_2 = 0$$

$$6c_1 + 2c_2 - 2c_3 = 0$$

$$2c_2 + c_3 - 2c_4 = 0$$

which solves $c_1 = -\frac{1}{2}$, $c_2 = -3$, $c_3 = -\frac{9}{2}$ and $c_4 = -21$. Thus we have a particular solution,

$$y_p = -\frac{1}{2}t^3 - 3t^2 - \frac{9}{2}t - 21$$

And our general solution to the (13) is

$$y = y_c + y_p = A_1e^t + A_2e^{-2t} - \frac{1}{2}t^3 - 3t^2 - \frac{9}{2}t - 21$$

I guess this method is not much of different from what you had learned in your high mathematics course, the crucial point is how you set up the function form of particular solution.

Let's look at another example,

$$y'' - 4y = te^t + \cos(2t) \tag{14}$$

Again, complementary solution first, characteristic equation is

$$r^2 - 4 = 0$$

with roots $r_1 = 2$ and $r_2 = -2$. Complementary solution is

$$y_c = A_1e^{2t} + A_2e^{-2t}$$

As to the particular solution, this is much trickier, we need to set up two particular solution, one is $y_{p_1} = (c_1t + c_2)e^t$ and $y_{p_2} = c_3\cos(2t) + c_4\sin(2t)$.

List their first and second derivatives,

$$
\begin{aligned}
y'_{p_1} &= (c_1 t + c_1 + c_2)e^t \\
y''_{p_1} &= (c_1 t + 2c_1 + c_2)e^t \\
y'_{p_2} &= -2c_3 \sin(2t) + 2c_4 \cos(2t) \\
y''_{p_2} &= -4c_3 \cos(2t) - 4c_4 \sin(2t)
\end{aligned}
$$

Rest of work is to substitute in and solve linear equation, but the general solution of (14) is

$$y = y_c + y_{p_1} + y_{p_2}$$

So the we omit the calculation here because the idea of the example is to teach how to handle two seperate variable terms.

2.2.2 Method of Variational Parameters

We only give a general case here, since all practical problem can follow this template. We have a differential equation

$$y'' + p(t)y' + q(t)y = g(t) \tag{15}$$

$p(t)$, $q(t)$ and $g(t)$ are continuous functions of t. Its complementary solution is

$$y_c = c_1 y_1(t) + c_2 y_2(t)$$

As usual, c_1 and c_2 are constants, they can be definitize by initial condition. Here is the how we start the method of variational parameters, we determine that the particular solution has a form of

$$y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$$

we change the constant coefficients into variables, but v_1 and v_2 are variables yet to be determined, and they are also the functions of t. We take derivative of y_p,

$$y'_p = v'_1 y_1 + v_1 y'_1 + v'_2 y_2 + v_2 y'_2$$

But we have make some assumption here, which is

$$v_1'y_1 + v_2'y_2 = 0$$

Then we reduce y_p' into,

$$y_p' = v_1y_1' + v_2y_2'$$

Take derivative again,

$$y_p'' = v_1'y_1' + v_2'y_2' + v_1y_1'' + v_2y_2''$$

Substitute into (15),

$$v_1'y_1' + v_2'y_2' + v_1y_1'' + v_2y_2'' + p(t)(v_1y_1' + v_2y_2') + q(t)(v_1y_1 + v_2y_2)$$

Rearrange, it equals,

$$v_1(y_1'' + p(t)y_1' + q(t)y_1) + v_2(y_2'' + p(t)y_2' + q(t)y_2) + v_1'y_1' + v_2'y_2'$$

Becasue y_1 and y_2 are solutions of complementary equation, so it is reduced to

$$y_p'' + p(t)y_p' + q(t)y_p = v_1'y_1' + v_2'y_2' = g(t)$$

We set up a linear equation system, one of them is the assumption we have made,

$$v_1'y_1' + v_2'y_2' = g(t)$$
$$v_1'y_1 + v_2'y_2 = 0$$

or in matrix form,

$$\begin{bmatrix} y_1' & y_2' \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} g(t) \\ 0 \end{bmatrix}$$

Solve it, we have

$$v_1' = \frac{y_2 g}{y_1y_2' - y_1'y_2}$$
$$v_2' = \frac{y_1 g}{y_1y_2' - y_1'y_2}$$

Final step, integrate them,

$$v_1 = \int \frac{y_2 g}{y_1 y_2' - y_1' y_2} \,\mathrm{d}t$$
$$v_2 = \int \frac{y_1 g}{y_1 y_2' - y_1' y_2} \,\mathrm{d}t$$

Substitute them back to $y_p = v_1 y_1 + v_2 y_2$,

$$y_p = \int \frac{y_2 g}{y_1 y_2' - y_1' y_2} \,\mathrm{d}t \, y_1 + \int \frac{y_1 g}{y_1 y_2' - y_1' y_2} \,\mathrm{d}t \, y_1$$

3 The Laplace Transform

This section will tell you how to use the *Laplace transform* to solve differential equation. The same as *Fourier transform*, it will turn a differential equation into an algebraic equation which of course is much easier to handle. Till this moment, all differential equations we study are continuous functions and their continuous derivatives. However, sometimes we really don't have such luxury. So the power of the Laplace transform is to handle strange-shaped functions, such as jumps, spikes, impulses and etc. So you can imagine the Laplace transform as a moving average filter, which smoothen all strange kinks or spikes. However this section is only teaching you how to solve differential equations with the help of the Laplace transform, but not Laplace transform itself. You should be prepared this knowledge when studying Fourier transform.[9].

4 Linear System of Differential Equations

Here we will make a small step ahead, we are dealing with a linear system here rather than a single varible differential equation. It would be more

[9] I know sometimes I ask too much, but if you know nothing about Fourier transform and Laplace transform, study my notes *Fourier Analysis*[3] and *Laplace Transform*[2].

realistic if we could model a system of varibles together, let them interact with each other. However we will only focus on the first-order differential equation system, any n^{th}-order single differential equation can be reduced down to a system of first-order differential equations, which means even if you have a system of n^{th}-order differential equations, you can always reduce them down one-by-one into a larger system of first-order differential equations.

A simple example might tell you how we perform this task, we use a simple physics model here,

$$s'' = f(s', s, t) \tag{16}$$

where s is displacement of any vehicle, s' is velocity, s'' is acceleration and t represents time. The key idea is to introduce a new variable to reduce the system down, v, velocity. Then we will have a system here,

$$\begin{aligned} v' &= f(v, s, t) \\ s' &= v \end{aligned}$$

As you can note easily, if $s' = v$ holds, the whole system will be equivalent to (16).

4.1 Solve Linear System

The general linear system of differential equation is

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$$

where

$$\boldsymbol{x}' = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} \qquad \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Some linear algebra knowledge will be made use of here. We have a first question here to ask ourselves, what if $\boldsymbol{A}$ is a diagonal matrix. Then the system will look like,

$$\begin{aligned} x_1' &= a_{11}x_1 \\ x_2' &= a_{22}x_2 \\ &\vdots \\ x_n' &= a_{nn}x_n \end{aligned}$$

This is not a linear system any more, they are just an array of single linear differential equations, you can randomly pick anyone to solve without information provided by other equations. If every linear system would be changed into this form, or should I say explicitly, if any matrix $\boldsymbol{A}$ could be transformed into a diagonal matrix $\boldsymbol{D}$, we could solve the system quickly without resort to any knowledge of linear equation system.

Knowledges are always entangled together, the more you learned mathematics the faster progress you will make. As you have guessed, what we are going to do here is diagonalization.

$$\boldsymbol{AP} = \boldsymbol{PD}$$

where

$$\boldsymbol{P} = [\boldsymbol{v}_1 \dots \boldsymbol{v}_n] \qquad \boldsymbol{D} = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}$$

$\boldsymbol{v}_i$ is eigenvector of $\boldsymbol{A}$ whereas λ_i is corresponding eigenvalue. Then the diagonalization will be,

$$\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{D}$$

Let's come back to our differential equation, with change of variable $\boldsymbol{y} =$

$\boldsymbol{P}^{-1}\boldsymbol{x}$,

$$\begin{aligned}\boldsymbol{y'} &= \boldsymbol{P}^{-1}\boldsymbol{x'}\\ &= \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{x}\\ &= \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{y}\\ &= \boldsymbol{D}\boldsymbol{y}\end{aligned}$$

Then the whole system is reduced down to an array of self-contained equations.

$$\begin{aligned}y_1' &= \lambda_1 y_1\\ &\vdots\\ y_n' &= \lambda_n y_n\end{aligned}$$

Now we can use previous knowledge of first-order differential equations, all of them has its own characteristic equation $r_i - \lambda_i = 0$, thus every homogenenous equation has solution has following,

$$\begin{aligned}y_1 &= c_1 e^{\lambda_1 t}\\ y_2 &= c_2 e^{\lambda_2 t}\\ &\vdots\\ y_n &= c_n e^{\lambda_n t}\end{aligned}$$

However, we need to change the variables back to $\boldsymbol{x}$,

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y}$$

It is

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

$$= c_1 e^{\lambda_1 t} \boldsymbol{v}_1 + c_2 e^{\lambda_2 t} \boldsymbol{v}_2 + \ldots + c_n e^{\lambda_n t} \boldsymbol{v}_n$$

$\boldsymbol{x}$ is a linear combination of all $\boldsymbol{A}$'s eigenvectors. As long as we have n linearly independent eigenvectors, we have use diagonalization to solve the equation system. Recall that if you have n distinct eigenvalues, there will be exactly n linearly independent eigenvectors. However when you have n linearly independent eigenvectors, you can't be sure that there will n distinct eigenvectors. For details refer to my notes of *Linear Algebra and Matrix Analysis I* [4].

References

[1] Chen W. (2011): *Complex Numbers*, study notes

[2] Chen W. (2011): *Fourier Analysis*, study notes

[3] Chen W. (2011): *Laplace Transform*, study notes

[4] Chen W. (2011): *Linear Algebra and Matrix Analysis I*, study notes

[5] Chiang A.C. and Wainwright K.(2005): *Fundamental Methods of Mathematical Economics*, McGraw-Hill

[6] Polking J., Boggess A. and Arnold D.(2006): *Differential Equations*, Pearson Presntice Hall

[7] Stewart J.(2003): *Calculus:Early Transcendentals*, Thomson Press

[8] Simon C. and Blume L.(1994): *Mathematics For Economists*, W.W.Norton & Company