P1

$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t$

$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)\, y_t = w_t$.

$1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0$

(1) multiply by z^{-p}

(2) Set $z^{-1} = \lambda$.

$\Rightarrow \lambda^p - \phi_1 \lambda^{p-1} - \cdots - \phi_p = 0$.

stable $\Leftrightarrow$ z outside unit circle $\Leftrightarrow$ λ inside unit circle.

$y_t = (1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)^{-1} w_t$.

$y_t = \frac{c_1}{(1-\lambda_1 L)} w_t + \frac{c_2}{(1-\lambda_2 L)} w_t + \cdots + \frac{c_p}{(1-\lambda_p L)} w_t$.

$c_i = \frac{\lambda_i^{p-1}}{\prod_{\substack{j=1 \\ j \neq k}}^{p} (\lambda_j - \lambda_k)}$

$\frac{\partial y_t}{\partial w_0} = c_1 \lambda_1^t + c_2 \lambda_2^t + \cdots + c_p \lambda_p^t$ $\quad \sum_{i=1}^{p} c_i = 1$

$\{y_1^{(i)}, y_2^{(i)}, \ldots, y_T^{(i)}\}$ $\quad \{Y_1, Y_2, \ldots Y_T\}$

$f(y_t)$

Example: if $Y_t \sim N(0, \sigma^2)$. $\Rightarrow f(y_t) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2\sigma^2} y_t^2\right\}$

$\mu_t = E(Y_t) = \int_{-\infty}^{+\infty} y_t f(y_t)\, dy_t$ $\quad$ Unconditional mean

$\gamma_{0,t} = E(Y_t - \mu_t)^2 = \int_{-\infty}^{+\infty} (y_t - \mu)^2 f(y_t)\, dy_t$ $\quad$ Unconditional Varian

P2.

$$\begin{bmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-j} \end{bmatrix} \quad \gamma_{jt} = E[(Y_t-\mu_t)(Y_{t-j}-\mu_{t-j})]$$

$$= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (y_t-\mu_t)(y_{t-j}-\mu_{t-j}) \cdot f(y_t, y_{t-1}, \dots, y_{t-j})\, dy_t \cdots dy_{t-j}$$

$$f(y_t) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{1}{2\sigma^2} \cdot y_t^2 \right\}$$

$$f(y_t, y_{t-1}, \dots, y_{t-j}) = \prod_{i=0}^{j} f(y_{t-i})$$

- $E(y_t) = \mu \quad \forall t.$
- $E(Y_t-\mu)^2 \equiv \gamma_0 \quad \forall t.$
- $E[(Y_t-\mu)(Y_{t-j}-\mu)] = \gamma_j \quad \forall t$

Covariance Stationary. (Weakly)

$f(y_t, y_{t-1}, \dots, y_{t-j})$ depends only on j. (strict stationary).

Covariance stationary $\not\Rightarrow$ strict stationary
$\Leftarrow$

Ergodicity $\bar{y} = \frac{1}{T}\sum_{t=1}^{T} y_t \xrightarrow{P} E(Y_t) = \mu.$

$$\hat{\gamma}_j = \frac{1}{T-j}\sum_{t=1}^{T-j} (y_{t+j}-\bar{y})(y_t-\bar{y}) \xrightarrow{P} \gamma_j$$

If $\sum_{j=-\infty}^{+\infty} |\gamma_j| < \infty$, then $\{Y_t\}$ is ergodic for the mean

If Y_t is stationary Guassian Process, then egodicty for all moments.

~~$Y_t-\mu_t$~~ Auto Regressive Moving Average Process (ARMA).

White noise: ε_t.

$E(\varepsilon_t) = 0$

$E(\varepsilon_t^2) = \sigma^2 \quad E(\varepsilon_t \varepsilon_{t-j}) = 0, \ \forall j$

P1

$Y_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$, ε_t is white noise.

$E(Y_t) = \mu$

$\gamma_0 = (1+\theta^2)\sigma^2$

$\gamma_1 = \theta \sigma^2$

$\gamma_j = 0, \ \forall j > 1$

$$Corr(Y_t, Y_{t-j}) = \frac{Cov(Y_t, Y_{t-j})}{\sqrt{Var(Y_t)}\sqrt{Var(Y_{t-j})}}$$

$$= \frac{\gamma_j}{\gamma_0} = \rho_j = \begin{cases} \frac{\theta}{1+\theta^2}, & j=1 \\ 0, & j>1 \\ 1, & j=0 \end{cases}$$

Autocorrelation function

MA(q)

$Y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_q \varepsilon_{t-q}$. ε_t is white noise $(0, \sigma^2)$

$E(Y_t) = \mu$, $Var(Y_t) = E(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q})^2$

$= (1 + \theta_1^2 + \dots + \theta_q^2)\sigma^2 = \gamma_0$

$Cov(Y_t, Y_{t-j}) = \gamma_j = E(\varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q})$

$(\varepsilon_{t-j} + \theta_1 \varepsilon_{t-j-1} + \theta_q \varepsilon_{t-j-q})$

$= (\theta_j + \theta_{j+1}\theta_1 + \theta_q \theta_{q-j})\sigma^2$

$$\gamma_j = \begin{cases} (1+\theta_1^2 + \dots + \theta_q^2)\sigma^2 & \text{if } j=0 \\ (\theta_j + \theta_{j+1}\theta_1 + \dots + \theta_q \theta_{q-j})\sigma^2 & \text{if } 1 \le j \le q \\ 0 & \text{if } j > q \end{cases}$$

$\sum_{j=0}^{\infty} |\gamma_j| < \infty$

$Y_t = \mu + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j}$ with $\theta_0 = 1$

MA(∞) $Y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$, with $\psi_0 = 1$

$E(Y_t) = E(\lim_{T\to\infty} \mu + \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \dots + \psi_T \varepsilon_{t-T})$

Assume that $\sum_{j=0}^{\infty} |\psi_j| < \infty \Rightarrow \sum_{j=0}^{\infty} \psi_j^2 < \infty$

P2

So $E(Y_t) = \lim_{T\to\infty} E(\mu + \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \cdots + \psi_T \varepsilon_{t-T})$

$= \mu.$

$\gamma_0 = E\left(\sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}\right)^2 = \sigma^2 \sum \psi_j^2 < \infty.$

$\gamma_j = E\left(\sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k}\right)\left(\sum_{k=0}^{\infty} \psi_k \varepsilon_{t-j-k}\right).$

$= \sigma^2 \sum_{k=0}^{\infty} \psi_{j+k} \psi_k < \infty.$

AR(1): $Y_t = c + \phi Y_{t-1} + \varepsilon_t$, $\varepsilon_t \sim$ white noise.

$(1-\phi L) Y_t = c + \varepsilon_t.$

If $|\phi| < 1$, $Y_t = (1-\phi L)^{-1}(c + \varepsilon_t).$

$= (c + \varepsilon_t) + \phi(c + \varepsilon_{t-1}) + \phi^2(c + \varepsilon_{t-2})$

$+ \cdots$

$= \frac{c}{1-\phi} + \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$

$= \frac{c}{1-\phi} + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \quad (\psi_j = \phi^j)$

$E(Y_t) = \frac{c}{1-\phi} = \mu$

$\gamma_0 = E(Y_t - \mu)^2 = (\varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + \cdots)^2$

$= \cancel{(1+\phi^2+} \frac{\sigma^2}{1-\phi^2}$

$\sum_{j=0}^{\infty} |\psi|^j = \frac{1}{1-|\psi|}$

$\gamma_j = E((Y_t - \mu)(Y_{t-j} - \mu))$

$= E(\varepsilon_t + \phi \varepsilon_{t-1} + \cdots + \phi^j \varepsilon_{t-j} + \cdots)$

$(\varepsilon_{t-j} + \phi \varepsilon_{t-j-1} + \cdots) = \sigma^2(\phi^j + \phi^{j+2} + \cdots) = \frac{\sigma^2 \phi^j}{1-\phi^2}$

P3

$$\rho_j = \begin{cases} 1 & \text{if } j=0 \\ \phi^j & \text{if } j \geq 1 \end{cases}$$

$Y_t = c + \phi Y_{t-1} + \varepsilon_t$

$E(Y_t) = E(c + \phi Y_{t-1} + \varepsilon_t)$

$\Rightarrow E(Y_t) = E(c) + \phi E(Y_{t-1}) + E(\varepsilon_t)$

$\Rightarrow E(Y_t) = \frac{c}{1-\phi} = \mu.$

$Y_t = c + \phi Y_{t-1} + \varepsilon_t \quad \mu = \frac{c}{1-\phi} \Rightarrow c = \mu(1-\phi)$

$Y_t = \mu(1-\phi) + \phi Y_{t-1} + \varepsilon_t.$

$Y_t - \mu = \phi (Y_{t-1} - \mu) + \varepsilon_t.$

$E(Y_t - \mu)^2 = E(\phi(Y_{t-1} - \mu) + \varepsilon_t)^2$

$= \phi^2 E(Y_{t-1} - \mu)^2 + E(\varepsilon_t^2)$

$\Rightarrow E(Y_t - \mu)^2 = \frac{\sigma^2}{1-\phi^2}$

$(Y_t - \mu)(Y_{t-j} - \mu) = \phi(Y_{t-1} - \mu)(Y_{t-j} - \mu) + \varepsilon_t (Y_{t-j} - \mu).$

$\Rightarrow \gamma_j = \phi \gamma_{j-1}$

$\Rightarrow \gamma_j = \phi^j \gamma_0 = \phi^j \frac{\sigma^2}{1-\phi^2}$

$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t$

$(1 - \phi_1 L - \phi_2 L^2) Y_t = c + \varepsilon_t.$

$Y_t = (1 - \phi_1 L - \phi_2 L^2)^{-1} (c + \varepsilon_t)$

$= \psi(L)(c + \varepsilon_t) = \psi(L) c + \psi(L) \varepsilon_t = \frac{c}{1-\phi_1-\phi_2} + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$

μ

$\psi_j = \frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_1^j + \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_2^j$

$+ \psi(L) \varepsilon_t$

P4. $\gamma_0, \cdots \gamma_j$

$\Rightarrow$ Yule-Walker Equations

$$Y_t = \mu(1-\phi_1-\phi_2) + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t.$$

$$(Y_t-\mu) = \phi_1(Y_{t-1}-\mu) + \phi_2(Y_{t-2}-\mu) + \varepsilon_t.$$

$$E(Y_t-\mu)^2 = \phi_1 E(Y_{t-1}-\mu)(Y_t-\mu) + \phi_2 E(Y_{t-2}-\mu)(Y_t-\mu) + E(\varepsilon_t)(Y_t-\mu).$$

$$\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2 \quad (*)$$

$$E(Y_t-\mu)(Y_{t-j}-\mu) = \phi_1 E(Y_{t-1}-\mu)(Y_{t-j}-\mu) + \phi_2 E(Y_{t-2}-\mu)(Y_{t-j}-\mu) + \varepsilon_t(Y_{t-j}-\mu)$$

$$\Rightarrow \gamma_j = \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2}.$$

$$\Rightarrow \rho_j = \phi_1\rho_{j-1} + \phi_2\rho_{j-2} \quad (**).$$

~~$\gamma_0 = \phi_1\gamma_1 + \phi$~~

$$\begin{cases} \gamma_0 = \phi_1\gamma_0\rho_1 + \phi_2\gamma_0\rho_2 + \sigma^2. \\ \rho_j = \phi_1\rho_{j-1} + \phi_2\rho_{j-2} \quad \forall j \end{cases}$$

$$\sum_{j=-\infty}^{+\infty} |\gamma_j| = \sigma^2 < \infty$$

MA(1). : $Y_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$

$$E(Y_t) = \mu.$$

$$E(Y_t - \mu)^2 = (1+\theta^2)\sigma^2.$$

$$\gamma_1 = E(Y_t - \mu)(Y_{t-1} - \mu) = E(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-1} + \theta\varepsilon_{t-2})$$

$$= \theta\sigma^2.$$

$$\gamma_j = 0 \quad (\forall j > 1).$$

$$E(Y_t) = \mu. \quad \gamma_j = \begin{cases} (1+\theta^2)\sigma^2, & \text{if } j = 0 \\ \theta\sigma^2 & \text{if } j = 1 \\ 0, & \text{if } j > 1. \end{cases}$$

$$\sum_{j=0}^{+\infty} |\gamma_j| = |\gamma_0| + |\gamma_1| \cancel{+ |\gamma_2|}$$

$$= (1 + |\theta| + \theta^2)\sigma^2$$

$\Gamma_1 + \Gamma_2$

$\Gamma_2 + \Gamma_3$

$\Gamma_3 + \Gamma_4$

$\vdots$