Endogenous fertility, education selection and economic growth
A theoretical approach explaining the difference between the pre-industrial and post-industriral revolution period

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1 A Brief Introduction

The main purpose of economic development is to figure out how the economy evolves through history. Industial revolution is the most pivotal event in economic history. After this particular point, the GDP per capital grows exponientially while remains at a constant level before. It is really challenging task to figure out how the economy makes progress before and after industrial revolution. People choose between receiving education, breeding offsprings and working during their adulthood life given particular social condition. The lifestyle of ordinary people could be significantly different before the technology progress. It leaves us great space to discuss the optimum personal choice between the education level and fertility rate under different aggregate output level. The fertility rate besides the education level is an important issue related to the economic growth theory and it should be carefully taken into account in economic development model so as to explain the evolution of the population growth endogenously. The industrial revolution, without any doubt, contributes the most important components of our nowadays happiness and convenivence. A considerable amount of machines are invented during this period to apparently increase the production efficiency and change the way of sharing knowledge through education. The production efficiency level has to be treated seriously in the economic progress explanation approach.

The paper is thus organized in the following structure to discuss such difference of individual personal choice before and after industrial revolution. First we develop a general three period overlapping generation model which a representative agent choose among raising kids, going to school and working for producing output. Then we discuss the optimal choices under different technology development level by making the exogeous assumption that marginal contribution of education to output level is a constant in one period. Some discussion is placed in the following sections about the difference between pre-industrial and post-industrial revolution period. After that we begin to exdogenizing all parameters to explain the major reasons of such difference. At the end of paper the conclusion is made and some open topics are remained to the future research.

2 Three Period Overlapping Generation Model

We consider an overlapping generation mode in a economy with a representative agent. The represented agent's total utiltiy depends on his consumption and reputation. In this article we suppose the agent lives through childhood, adult and the elder age. All the prodution are produced by people who are adulthood currently. We assume no government involvement in our model. As a result, we can never use fiscal policy to collect tax revenue from each family and redistribute to the society in a centain form.

In the childhood, yound kids live completely on their parents. In the model, denote y_t as the to total output of each parent. Each parent spends $\theta_t/2$ of their output to raise their kids. As a result one kid can consume $y_t\theta_t$ during his childhood life. $C_{1,t}$ is used to represent childhood comsuption when the agent is a child in period t.

Then the represent agent grows up from childhood to adultlhood. We suppuse a survival rate π_1, which means the representative agent has a chance of $1-\pi_1$ that he cannot survive the childhood age. Here we make the general assumption that $0<\pi_1<1$. At the beginning of the calculation of the model, it is supposed to be true that π_1 is exogenous. And in the later part of the article π_1 will be endogenized. The representative agent chooses whether to spend some time receiving education or go to work directly. If the producer decides to go to school he/she will lose the opportunity chance to produce ouput. We assume the education level has a non-negative effect on the production efficiency. With different social aggregate production level, the education level exhibits an increasing marginal production efficiency effect under larger aggregate output level. Also during adulthood life the agent needs to decide how many offsprings to raise. The reason of having children is quite natural since human being, as a species of mammal, wishes to make the gene live forever by raising kids. In our model, we assume during pregnancy and lactation period, the represented agent cannot work to increase their utility level. During this period, parents have to devote themselves thoroughly in the birth to a new life. In the model I make a reasonable assumption that an adult cannot breed children during the time period he/she goes to school. It is naturally the case particularly in recent centuries because the education becomes even more complex and therefore an adult can seldomly breed children and go to school for study simultaneously.

During the adulthood life, besides an investment in raising kid, the representative agent also need to pay their parents pension expenses as there is no government involvement in this model. That is to say, if parents of the agent are still alive, there should exist a portion deduction of the total ouput of this represetative agent to raise parents since parents in their elder age are retired and do not make further prodution.

There is a probably π_2 that an adulthood can survive to his/her old age. It is quite similar to π_1 case that we may assume $0<\pi_2<1$ and make it exogenous at the begining of the discussion process. Later on it shall be endogelized and discussed carefully. When a person becomes older he/she stops working and lives only on his/her pension from their children. In this particurlar model the pension is collected from the next generation of the family. Parents raise kids when they are young and lives on their grown-up children when older and becomes no longer suitable for work.

A yound parent chooses to maximize his/her lifetime utility as follows:

$$U=[\frac{1}{1-\sigma}][C_{1,t}^{1-\sigma}-1]+\beta\pi_1[\frac{1}{1-\sigma}][C_{2,t+1}^{1-\sigma}-1]+\beta^2\pi_1\pi_2[\frac{1}{1-\sigma}][C_{3,t+2}^{1-\sigma}-1] \quad (1)$$

subject to

$$C_{1,t}=\theta_t y_t \quad (2)$$

$$y_t=(e_t^{\alpha_e}+\bar{l}^{\alpha})^{\frac{1}{\alpha}}l_t \quad (3)$$

$$e_{t+1} + mn_{t+1} \leq L_1 \tag{4}$$

$$e_{t+1} + mn_{t+1} + l_{t+1} \leq L_2 \tag{5}$$

$$\pi_1 = \pi_1(Y_t) \tag{6}$$

$$\pi_2 = \pi_2(Y_t) \tag{7}$$

$$\alpha_e = \alpha_e(Y_t) \tag{8}$$

$$Y_t = N_t y_t \tag{9}$$

$$C_{2,t+1} = f_{t+1} w_{t+1} l_{t+1} \tag{10}$$

$$f_{t+1} = 1 - \frac{1}{2} n_{t+1} \theta_{t+1} - \pi_2 x_{t+1} \tag{11}$$

$$C_{3,t+2} = \frac{1}{2} \pi_1 x_{t+2} n_{t+1} C_{2,t+2} \tag{12}$$

The choice variables here are $\{n_t\}_{t=0}^{\infty}$,$\{\theta_t\}_{t=0}^{\infty}$,$\{x_t\}_{t=0}^{\infty}$, $\{e_t\}_{t=0}^{\infty}$, $\{l_t\}_{t=0}^{\infty}$

Since the wage rate are competitively determined we have

$$w_t = \frac{\partial}{\partial l_t} (e_t^{\alpha_e} + \bar{l}^{\alpha})^{\frac{1}{\alpha}} l_t = (e_t^{\alpha_e} + \bar{l}^{\alpha})^{\frac{1}{\alpha}} \tag{13}$$

In the expression (1), we assume a constant relative risk aversion utility function with respect to consumption in different periods. β denotes the time perference. We makes no assumption on β. A representative agent might have $\beta > 1$ just because he/she is more concerned with consumption in old age while in some other cases he/she may have $\beta < 1$ since childhood life is the most important period in this case.

In equation (2), θ_t is the portion of parent's income spending on each kid. In out notation, $C_{1,t} = \theta_t y_t$ is the total consumption in the agent's childhood life. The agent can only live on his/her parents' income when he/she is in the young age.

In equation (3), l_t is the total time the representative producer chooses to produce in his life. It is generally assumed that the production efficiently depends on both e_t which is education length and $\bar{l}$ which is the raw labor. If a person has never gone to school he can still make production through his raw labor successfully such as conducting irregation, hunting and doing laundry jobs. Here CES production function is used with parameter α. An additional parameter α_e denotes the contribution level of education to the production. We may first assume α_e is an exogenous variable and then exdogenize it among historical evolution.

In constraint (4), L_1 denotes the maximum year that a consumer can breed offsprings and go to school and n_t is the number of children(i.e. the fertility rate). Here m is the parameter representing pregnancy and breed length of a single child. This parameter is obviously longer than 10 months but in more general cases it can be a period of one to two years since a person must devote almost all his/her time and energy to the baby when the child is very young. It is well known that woman can hardly give birth to a child above a centain age. This is the most pivotal point in this inequality constrant and we will see it plays an important role in the fertility rate decision making process in the Middle Ages.

In constraint (5), L_2 denotes the length of adulthood life. The producer chooses during his adulthood how to seperate his/her time into going for education, raising kids and working for outputs. It is quite obvious the inequality constraint wil become an equality constraints when doing optimization. The reason is that if the constraint is slack the agent can always increase the production time to increase his second period's utility level. Certainly we have $L_2 > L_1$.

Through equations (6) to (9), three parameters π_1, π_2 and α_e are introduced, where π_1 is the survival rate from childhood to adult,π_2 defines the rate the agent has the luck to retire and α_e denotes the contribution of education to producing. These three parameter are supposed to be exogenous at the begining of the calculcation but later on we will endogelize all these three by using equation (9). In that case, it is assumed the surviving rates depend on the aggregate social output level. Also the increasing of GDP will result in dramatical change in the parameter α_e, which leads a completely different equilibrium of the model. It shall be seriously taken into account the rapid change of the importance of education level among history that leads to the great improvement of production efficiency. From this model I make a trial to illustrate the evolving process of industrial revolution by making some assumptions on α_e. N_t defines total population of a generation.

In equation (10) and (11), $C_{2,t+1}$ is the consumption of an adult in period $t+1$. In this three-period overlapping generation model we assume that the total consumption is a fraction of output. Each adult has to spend $\frac{1}{2}\theta_t$ of total output to raise each single child. In the model x_t denotes the proportion that the consumer will give the output to parents if they are still alive. The expectation of such propotion is $\pi_2 x_t$. As no government is included in this model, it ensures the agent will have the basic necessities of life when having no ability to work anymore by receiving money from their children.

In equation (12), $C_{3,t+2}$ defines the consumption of an old person in period $t+2$. According to the previous explanation, the component of $C_{3,t+2}$ depends on the tranferring payments from offsprings. The expected value of this part of consumption is the product of survival rate of the children, the number of kids, the output each child will transfer.

When we eliminate $\{l_t\}_{t=0}^{\infty}$ we have to solve the following problem:

$$\begin{aligned}
\max_{\{n_t,\theta_t,x_t,e_t\}_{t=0}^{\infty}} U &= [\frac{1}{1-\sigma}((\theta_t(e_t^{\alpha_e}+\bar{l}^{\alpha})^{\frac{1}{\alpha}}(L_2-e_t-mn_t))^{1-\sigma}-1)] \\
&+\beta\pi_1\frac{1}{1-\sigma}[((e_{t+1}^{\alpha_e}+\bar{l}^{\alpha})^{\frac{1}{\alpha}}(L_2-e_{t+1}-mn_{t+1})(1-\frac{1}{2}n_{t+1}\theta_{t+1}-\pi_2 x_{t+1}))^{1-\sigma}-1] \\
&+\beta^2\pi_1\pi_2\frac{1}{1-\sigma}[[\frac{1}{2}x_{t+2}\pi_1 n_{t+1}(e_{t+2}^{\alpha_e}+\bar{l}^{\alpha})^{\frac{1}{\alpha}}(L_2-e_{t+2}-mn_{t+2})]^{1-\sigma}-1]
\end{aligned} \tag{14}$$

3 Pre-industrial revolution Period Equilibrium and Economic Growth

We devide the economic history into two sections: the pre-industrial revolution period and the post-industrial revolution period. Before the industrial revolution occurs, actually the level of education plays a weak role in daily production. Most people use their raw labor to cultivate the agricultural plants and manually build houses. Why is it the case before industrial revolution? That is because marginal contribution of education is far less than the raw labor. In that period the social aggregate output level is so low that the benefit of going to school for more education is approximately zero. We can only learn the literature, arts, philosophy when educated, which seem to have no improvement effect on production. No machines are created at that time so that the contribution of education can be ignored.

As a result, before the industrial revolution, we have $\alpha_e \to 0$. Hence, we reduce the model in pre-industrial revolution period into following situation by setting $\alpha_e = 0$.

From now on we will discuss some properties of the general equilibrium before industrial revolution period under reasonable assumptions. Some attempts will be made to try to illutrate how mankind evolves before the industrial revolution.

Proposition 1. Before the industrial revolution since the lack of importance of education, a representative will generally choose the the education level $e = 0$

Proof. In the utility function we calculate $\frac{\partial U}{\partial e_t}$,$\frac{\partial U}{\partial e_{t+1}}$ and $\frac{\partial U}{\partial e_{t+2}}$. We can have the following results without difficulties:

$$\frac{\partial U}{\partial e_t} < 0, \frac{\partial U}{\partial e_{t+1}} < 0, \frac{\partial U}{\partial e_{t+2}} < 0. \tag{15}$$

As a result, in each period, the producer will not choose to go to school. □

Remark. Before the industrial revolution, we take the exogenous parameter $\alpha_e \to 0$ as given, which indicates the education from school makes few contribution to production. In fact, before machines are invented in large-scale, the knowledge learned from school could hardly beat the efficiency of the raw labor during producing goods for necessities. This is because the low amount of population level could not lead to the scale effect of education. As a result, any rational individual chooses to make none investment in education. This is quite different from the situation after industrial revolution, during which the knowledge and high-tech play the most significant role in production.

Now the choice variables remain only $\{n_t\}_{t=0}^{\infty}$,$\{\theta_t\}_{t=0}^{\infty}$,$\{x_t\}_{t=0}^{\infty}$, $\{l_t\}_{t=0}^{\infty}$ The above model can be simplified as

$$\max_{\{n_t,\theta_t,x_t,k_t\}_{t=0}^{\infty}} [\frac{1}{1-\sigma}((\theta_t\bar{l}(L_2-mn_t))^{1-\sigma}-1)] \\ +\beta\pi_1\frac{1}{1-\sigma}[(\bar{l}(L_2-mn_{t+1})(1-\frac{1}{2}n_{t+1}\theta_{t+1}-\pi_2x_{t+1}))^{1-\sigma}-1] \\ +\beta^2\pi_1\pi_2\frac{1}{1-\sigma}[[\frac{1}{2}x_{t+2}\pi_1n_{t+1}\bar{l}(L_2-mn_{t+2})]^{1-\sigma}-1] \tag{16}$$

Proposition 2. Before the industrial revolution the equilibrium level of n of a representative agent is give by

$$n_e=\frac{L_1}{m} \tag{17}$$

if the time preference for the future β is significantly high.

Proof. Denote the equilibrium level of C_2, l,f,θ,x,n before the industrial revolution period as C_{2e}, l_e,f_e,θ_e,x_e,n_e. Since an adult at period $t+1$ can only make the decision of n_{t+1}, we ignore the fact of n_t and n_{t+2} since they cannot be controlled by the adult of this generation. $\frac{\partial U}{\partial n_{t+1}}$ is calculated and evaluated at the equilibrium.

$$\frac{\partial U}{\partial n_{t+1}}|_{C_2,I,f,\theta,x,n=C_{2e},I_e,f_e,\theta_e,x_e,n_e} \\ =\beta\pi_1C_{2e}^{-\sigma}\bar{l}(-\frac{1}{2}\theta_el_e-mf_e)+\beta^2\pi_1\pi_2(\frac{C_{2e}}{f_e}\frac{1}{2}x_e\pi_1n_e)^{-\sigma}\bar{l}(l_e\frac{1}{2}x_e\pi_1) \\ =\beta\pi_1\bar{l}C_{2e}^{\sigma}\left[-\frac{1}{2}\theta_el_e-mf_e+\beta\pi_2(\frac{1}{f_e}\frac{1}{2}x_e\pi_1n_e)^{-\sigma}(l_e\frac{1}{2}x_e\pi_1)\right] \tag{18}$$

When we set

$$\beta>\frac{\frac{1}{2}\theta_el_e+mf_e}{\pi_2(\frac{1}{f_e}\frac{1}{2}x_e\pi_1n_e)^{-\sigma}(l_e\frac{1}{2}x_e\pi_1)} \tag{19}$$

it is obviously that at this time $\frac{\partial U}{\partial n_{t+1}}>0$. Combining with the contraint (4) which limits the range of n_{t+1}

$$e_{t+1}+mn_{t+1}\leq L_1 \tag{20}$$

the maximum optimum children number satisfies $n_e=\frac{L_1}{m}$. □

Remark. In Middle Ages the fertility rate is significantly high. **Proposition 2.** is a kind of proof to show the reason of such high fertility rate. Under the poor medical treatment and the consequently high death rate, human beings have to raise as many children as they can to proliferate. If mankind choose low feriltiy rate at that time, we human beings might have been distinct and could no longer have survived till today. But the child-bearing period has the maximum length of L_1 so that there exists a upper limit bound of the fertility rate. This proposition clarifies an individual keeps giving birth to new babies without interval when they are at child-bearing period before the industrial revolution when the time preference factor is significantly high. β, in a certain extent, defines human being's altruism for next generation. Before the industrial revolution, such altruism is so high that the population of human beings could keep growing, having a chance embrace the sunrise of technology rapid growth aftercoming.

Proposition 3. The number of the live adults grows at a rate of $g = \frac{\pi_1 L_1}{2m} - 1$ for each generation. Denote N_t as the living adult population in period t. It is clearly that $N_t = (1+g)N_{t-1} = (\frac{\pi_1 L_1}{2m} - 1)N_{t-1}$. P_t is used to calculate the total population in period t. The population is calculated as follows:

$$\begin{aligned} P_t &= \frac{n_t}{2}N_t + N_t + \pi_2 N_{t-1} \\ &= (\frac{L_1}{2m} + 1 + \frac{\pi_2}{\frac{\pi_1 L_1}{2m} - 1})N_t \end{aligned} \tag{21}$$

Also

$$P_{t+1} = (\frac{L_1}{2m} + 1 + \frac{\pi_2}{\frac{\pi_1 L_1}{2m} - 1})N_{t+1} = (\frac{L_1}{2m} + 1 + \frac{\pi_2}{\frac{\pi_1 L_1}{2m} - 1})(1+g)N_t = (1+g)P_t \tag{22}$$

The growth rate of total population between periods is $\frac{\pi_1 L_1}{2m} - 1$. □

Proposition 4. In our three-period overlapping generation model, there is no output per capita growth before the industrial revolution. But the total aggregate output Y_t level grows at constant rate g bacause of the contant population growth.

$$Y_{t+1} = N_{t+1}y_{t+1} = (1+g)N_t y_t = (1+g)Y_t \tag{23}$$

□

4 Post-industrial revolution Period Equilibrium and Economic Growth

The situation becomes quite different after industrial revolution. At this period, the knowledge-driven technology plays a critical value in economic growth. In other words, the technology develops so fast that the marginal output contribution through education outweighs the marginal product of raw labor. During this period, it is no longer optimum choosing not to go to school since the education can greatly affect the output efficiency. It is discussed in the section how economic grows after the industrial revolution under our structure of three-period overlapping generation model. The representative agent no longer concerns only about the number offsprings but pays more attention on the his own consumption level when he/she lives in the second period of life.

In order to illustrate the model, firstly we should treat the parameter α_e as an exogenous value to discuss how a representative agent makes optimum education/breeding/working dicision under constant level of the educational contribution to the production. When we discuss the properties among the steady state it is assumed that e_e, which is the steady state of education, is much great than the raw labor contribution $\bar{l}$. Hence it is reasonable to assume $\alpha_e >> \bar{l}$.

Then the model turns out to be as the following:

$$\max_{\{n_t,\theta_t,x_t,e_t\}_{t=0}^{\infty}} U = [\frac{1}{1-\sigma}((\theta_t e_t^{\frac{\alpha_e}{\alpha}}(L_2 - e_t - mn_t))^{1-\sigma} - 1)]$$
$$+ \beta\pi_1 \frac{1}{1-\sigma}[(e_{t+1}^{\frac{\alpha_e}{\alpha}}(L_2 - e_{t+1} - mn_{t+1})(1 - \frac{1}{2}n_{t+1}\theta_{t+1} - \pi_2 x_{t+1}))^{1-\sigma} - 1] \quad (24)$$
$$+ \beta^2\pi_1\pi_2 \frac{1}{1-\sigma}[[\frac{1}{2}x_{t+2}\pi_1 n_{t+1} e_{t+2}^{\frac{\alpha_e}{\alpha}}(L_2 - e_{t+2} - mn_{t+2})]^{1-\sigma} - 1]$$

Proposition 5. After the industrial revolution, for period $t+1$, the representative agent's optimum education decision is given by

$$e_{t+1} = \min(\frac{L_2}{1 + m + \frac{\alpha}{\alpha_e}}, L_1) \quad (25)$$

Proof. $\frac{\partial U}{\partial e_{t+1}}$ is calculated and evaluated at the equilibrium since the education level e_{t+1} is selected by the representative agent when he is an adult.

$$\frac{\partial U}{\partial e_{t+1}}|_{C_2,I,f,\theta,x,n=C_{2,t+1},I_{t+1},f_{t+1},\theta_{t+1},x_{t+1},n_{t+1}}$$
$$= \beta\pi_1 C_{2,t+1}^{-\sigma} f_{t+1}\left[\frac{\alpha_e}{\alpha} e_{t+1}^{\frac{\alpha_e}{\alpha}-1}(L_2 - e_{t+1} - mn_{t+1}) - e_{t+1}^{\frac{\alpha_e}{\alpha}}\right] \quad (26)$$
$$= \beta\pi_1 C_{2,t+1}^{-\sigma} f_{t+1} \frac{\alpha_{t+1}}{\alpha} e_{t+1}^{\frac{\alpha_e}{\alpha}-1}\left[L_2 - e_{t+1} - mn_{t+1} - e_{t+1}\frac{\alpha}{\alpha_e}\right]$$

When we set

$$\frac{\partial U}{\partial e_{t+1}}|_{C_2,I,f,\theta,x,n=C_{2,t+1},I_{t+1},f_{t+1},\theta_{t+1},x_{t+1},n_{t+1}} = 0 \quad (27)$$

the equilibrium should be

$$e_{t+1} = \frac{L_2}{1 + m + \frac{\alpha}{\alpha_e}} \quad (28)$$

But the constraint (4) should be taken into account simultanenously. Consequently the following result is obtained:

$$e_{t+1} = \min(\frac{L_2}{1 + m + \frac{\alpha}{\alpha_e}}, L_1) \quad (29)$$

□

Proposition 6. When β is significantly large, in other words when

$$\beta > \frac{mC_2^{-\sigma}}{\frac{1}{2}\pi_1\pi_2 X C_3^{-\sigma}} \quad (30)$$

the agent chooses to raise children with number

$$n_{t+1} = \frac{L_1 - e_{t+1}}{m} \tag{31}$$

when he/she makes decision at period $t+1$. Note here e_{t+1} is the result gained from **Proposition 5.** Otherwise Under

$$\beta < \frac{mC_2^{-\sigma}}{\frac{1}{2}\pi_1\pi_2 x C_3^{-\sigma}} \tag{32}$$

the agent does not choose to raise any children. The optimum children number in period $t+1$ is 0.

Proof. When we take the derivative with respect to n_{t+1}:

$$\begin{aligned} &\frac{\partial U}{\partial n_{t+1}}|_{C_2,I,f,\theta,x,n=C_{2,t+1},I_{t+1},f_{t+1},\theta_{t+1},x_{t+1},n_{t+1}} \\ &= \beta\pi_1 C_{2,t+1}^{-\sigma} e_{t+1}^{\frac{\alpha_e}{\alpha}} f_{t+1}(-m) + \beta^2\pi_1\pi_2 C_{3,t+2}^{-\sigma} f_{t+2}\frac{1}{2}x_{t+2}\pi_1 e_{t+2}^{\frac{\alpha_e}{\alpha}} \end{aligned} \tag{33}$$

When we evaluate $f_{t+1} = f_{t+2} = f_e$ and $e_{t+1} = e_{t+2} = e_e$ in the abovementioned expression can gain the result:

$$\frac{\partial U}{\partial n_{t+1}} = \beta\pi_1 f_e e_e^{\frac{\alpha_e}{\alpha}}\left[-mC_{2,t+1}^{-\sigma} + \frac{1}{2}\beta\pi_1\pi_2 x_{t+2} C_{3,t+2}^{-\sigma}\right] \tag{34}$$

If $\beta > \frac{mC_2^{-\sigma}}{\frac{1}{2}\pi_1\pi_2 X C_3^{-\sigma}}$, then $\frac{\partial U}{\partial n_{t+1}} > 0$ we choose to raise as many kids as we can. But we should also consider the constraint (4). In this case we can get

$$n_{t+1} = \frac{L_1 - e_{t+1}}{m} \tag{35}$$

Othewise if $\beta < \frac{mC_2^{-\sigma}}{\frac{1}{2}\pi_1\pi_2 X C_3^{-\sigma}}$, then $\frac{\partial U}{\partial n_{t+1}} < 0$

In this case the agent prefers not to raise any kids. The model will crash when β is not significantly large. □

Remark. As we mention in the last section, the altruism to the next generation is the reason we human beings are still alive on the earth. When time preference to the next period β passes a threshold value, mankind could evolve to the next generation. However, from **Proposition 6.** when β is small enough, the agent's optimum choice is to have no kids. Recently we witness many couples in the society maintain the faith of "Double Income and No Kids",which indicates their altruism for next generation is so low that they do not want to have kids. Generally speaking, it can be concluded the preconditional love to offsprings is the reason that human being could evolve till nowadays.

5 Comparative Static Analysis

In this section we do some comparative static analysis to discuss the parameters which affect the decision of fertility and education. First we focus on the phenomenon of DINK(Double Income and No Kids). We should analyze which parameters affect the decision of becoming DINKs. From last section we know the threshold time preference level of having kids is $\beta_{threshold} = \frac{mC_2^{-\sigma}}{\frac{1}{2}\pi_1\pi_2 XC_3^{-\sigma}}$. Then the following conclusions could be attained.

Proposition 7.

$$\frac{\partial \beta_{threshold}}{\partial m} > 0, \quad \frac{\partial \beta_{threshold}}{\partial \pi_1} < 0, \quad \frac{\partial \beta_{threshold}}{\partial \pi_2} < 0,$$
$$\frac{\partial \beta_{threshold}}{\partial x} < 0, \quad \frac{\partial \beta_{threshold}}{\partial C_2} > 0, \quad \frac{\partial \beta_{threshold}}{\partial C_3} < 0 \tag{36}$$

Remark. These results are quite intuitive. When it takes more time the breed a child or the individual want to enjoy the adulthood life, they become unwilling to have kids. On the contrary, when the agent prefers to enjoy his old age, the importance of having kids becomes significantly crucial. In our model there is no place for government. Therefore kids play an indispensible role in their parents retiring life. DINKs have to sacrifice the consumption when they become old to better enjoy the adulthood life. Today's pain of raising kids is the gain in the next period. There is a tradeoff between the altruism to next generation and the happiness of adulthood life.

Proposition 8. Now we may assume $\frac{L_2}{1+m+\frac{\alpha}{\alpha_e}} < L_1$ since in most cases education displays a diminishing return (hence $\frac{\alpha}{\alpha_e} > 1$). In this case the optimum decision rule for education is $e_{t+1} = \frac{L_2}{1+m+\frac{\alpha}{\alpha_e}}$. It could be concluded by exerting comparitive static analysis on this result.

$$\frac{\partial e_{t+1}}{\partial m} < 0 \tag{37}$$

$$\frac{\partial e_{t+1}}{\partial \alpha_e} > 0 \tag{38}$$

Remark. Proposition 8. is quite intuitive. When it becomes more time consuming to raise kids, an individual chooses to receive less education. In our overlapping generation model a person can never study and give birth to a kid at the same moment. Because of the length limit of possible breeding period, a rational invidual chooses to quit the school in order to raise more kids. This phenonenon is frequently observed in recent decades, particularly after the introducion of Post-doc system. A quite amount of post-doctors quit the programs just because they might want to raise more kids or hunt for better jobs on the job market.

6 Endogenization Approach

In the previous sections the education contribution to the production function α_e is exogenously given. It is obviously observed that it plays an important role in the illustrating the difference between pre-industrial and post-industrial revolution period optimal choice. In this section we make an attempt to illustrate such difference by endoginizing α_e. It is quite natural to figure out what factors influence the growth pattern of α_e since it dominates the future economic growth path. We will select α_e carefully to make it consistent to the whole model.

Define the growth rate of the total output level is g_Y which indicates that $Y_{t+1} = (1 + g_Y)Y_t$. The growth rate of the adult number is specified as g_n. According to **Proposition 5** and assuming human being not stoping to proliferate, we have

$$g_n = \frac{L_2}{1 + m + \frac{\alpha}{\alpha_e}} \tag{39}$$

Also the growth rate of total population is defined by g_p which has the following form:

$$\begin{aligned} P_{t+1} &= (\frac{e_{t+1}}{2}N_{t+1} + N_{t+1} + \pi_2 N_t) \\ &= (1 + g_n)(\frac{e_t}{2}N_t + N_t + \pi_2 N_{t-1}) = (1 + g_n)P_t \end{aligned} \tag{40}$$

So the growth rate of the total population $g_p = g_n$, which is quite intuitive in our three period overlapping generation model.

Considering the growth rate of the aggregate output level and we may deduce the endogenous growth pattern of α_e. From equation (8) we already know that

$$\alpha_e = \alpha_e(Y_t) \tag{41}$$

So it could be calculated as :

$$Y_{t+1} = N_{t+1}y_{t+1} = (1 + g_n)N_t y_{t+1} \tag{42}$$

At the same time we have :

$$Y_{t+1} = (1 + g_Y)Y_t \tag{43}$$

and

$$\frac{y_{t+1}}{y_t} = \frac{e_e^{\frac{\alpha_e(Y_{t+1})}{\alpha}}}{e_e^{\frac{\alpha_e(Y_t)}{\alpha}}} \tag{44}$$

It could be solved that

$$1 + g_Y = (1 + g_n)e_e^{\frac{1}{\alpha}(\alpha_e(Y_{t+1}) - \alpha_e(Y_t))} \tag{45}$$

In other word, α_e could be expressed in difference equation form:

$$\alpha_e(Y_{t+1}) - \alpha_e(Y_t) = \alpha[ln(1 + g_Y) - ln(1 + g_n) - ln(e_e)] \tag{46}$$

We have the following result.

Proposition 9 The growth rate of $\alpha_e(Y_t)$ is a linear transformation of the logrithm growth rate of aggregate output level. In other words, if the total output level grows exponentially, the growth rate of the contribution of education is in linear form.

This is the most crucial point we have thus reached so far. It could be deduced from the total output equation the exact explicit expression of the contribution of the education level to output efficiency. We can conclude from **Proposition 9** under exponential growth rate g_Y of the aggregate output level after industrial revolution between generations, the contribution of knowledge grows far less than the exponential growth rate.

Right now we switch to the expresson of education level. Imagine under the rapid growth of α_e, the representative agent will choose to be educated more since

$$e_{t+1} = \min(\frac{L_2}{1+m+\frac{\alpha}{\alpha_e}}, L_1) \tag{47}$$

Combining with the breeding constraint

$$e_{t+1} + mn_{t+1} \leq L_1 \tag{48}$$

So when α_e reachs a siginificant high level, it shall be concluded that a rational individual will not give birth to any children in order to maximize his three period overall utility. But what does it mean? It means the end of the world after this generation that is not tolerated by the model. However, the phenomenon of DINK is widely observed in recent decades. It still remains an open question of human beings stoping raising children any more.

Also, the result of endenization approach indicates that the assumption of $\alpha_e = 0$ is consistent before the industrial revolution period. Since there is no exponiential growth of total output at that time, the contribution of education displays approximately no change.

Now it is the time to exdogenize two parameters $\pi_1(Y_t)$ and $\pi_2(Y_t)$ and discuss the comparative static analysis result of the endogenization approach of these two parameters. In this part, with the development of the economy through the history, we try to use exponiential distribution to endogenize these two parameters. Assume

$$\pi_1(Y_t) = 1 - \exp \lambda_1 Y_t \tag{49}$$

$$\pi_2(Y_t) = 1 - \exp \lambda_2 Y_t \tag{50}$$

Here since the survival rate from adult to old age is significantly lower than the survival rate from young age to adult. It is quite reasonable to assume $\lambda_1 > \lambda_2$.

It is noticed that the decision rule of e and n do not depend on π_1 and π_2. What we could do right now is try to figure out how the threshold value $\beta_{threshold}$ changes with respect to $\pi_1(Y_t)$ and $\pi_2(Y_t)$ after endogenization.

$$\frac{\partial \beta_{threshold}}{\partial Y_t} = \frac{\partial \beta_{threshold}}{\partial \pi_1}\frac{\partial \pi_1}{\partial Y_t} + \frac{\partial \beta_{threshold}}{\partial \pi_2}\frac{\partial \pi_2}{\partial Y_t} \tag{51}$$

According to **Proposition 9**

$$\frac{\partial \beta_{threshold}}{\partial \pi_1} < 0, \quad \frac{\partial \beta_{threshold}}{\partial \pi_2} < 0 \tag{52}$$

and

$$\frac{\pi_1}{\partial Y_t} > 0, \quad \frac{\pi_2}{\partial Y_t} > 0 \tag{53}$$

We have

$$\frac{\partial \beta_{threshold}}{\partial Y_t} < 0 \tag{54}$$

The result provides us hint that the development of economics decreases the death rate so that parents are more willing to raise kids to receive offsprings' payoffs during the third period of their lives with a greater probabilty. The altruism level goes down with the progress and development of economics. People rely on the large survival rate under technology progress and raise more kids after industrial revolution comparing to live simply on the ideal expectation of better life of next generation before. In this aspect it might be concluded people live more "rational" after the industrial rovolution.

7 Conclusional Remark

The objective of this paper is to illustrate the difference of the economic development process before and after the industrial revolution period. It could be concluded from the article that there is no output per capital growth before the indutrial revolution. The output growth growth is completely triggered by the population growth at that time. As the death rate is relatively high under low development level of the economy, an individual chooses to proliferate as many children as they can during adulthood life while selects zero education level since it helps nothing but a waste of time to producing outputs. The marginal contribution of education could not outweigh raw labor and consequently no one chooses to go to school. The growth rate of the economy is identical to the population growth rate.

After the industrial revolution as technology component plays an indispensible role in economic growth, a representative agent chooses between receiving education and raising kids during child-bearing period. A person with more altruism towards next generation tends to have more kids instead of receiving high level of education. The choice hence reduces the birth rate after industrial revolution but leads to rapid growth of output efficiency during this period. As a result the population growth starts to become steady while the output per capital increases sharply and dramatically. The aggregate output increases exponientially in the post-industrial revolution.

There remains two open questions in this paper. Firstly, according to the paper, the rapid growth of technology and aftercoming aggregate output level make people less altruistic to offsprings. An increasing number of people are no longer willing to give birth to children. How mankind society will evolve under such situation? The model and social regulation will crash if human beings stop producing next generation. Secondly, at the very begining of the industrial revolution there should be a mixed effect of education and raw labor contribution to the output level. However, in the paper it is only discussed the case that marginal contribution of education is much more efficient than raw labor. There remains a huge challenge how the economy evolves under the mixed effect of raw labor and educational training. It could be imagined the per capital growth rate should be a litte bit lower but the fertility rate could be higher than the situation when raw labor is completely ignored. The process has to be more complicated than our model. We are very delighted to see the refinement of our models and discussion of the evolution process in the transitional period.