Fourier Analysis Lite

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Updated 27 July, 2011

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Abstract

This notes introduces you Fourier series and Fourier transforms, which are vastly made use of in engineering, physics, mathematics and economics. Cursory complex number knowledge is required before any attempt to Fourier analysis[1], besides you need some basic knowledge of integral calculus and trigonometry.

1 Fourier Series

Fourier series is by no means as intuitive as, say Taylor series, in the very beginning. And notations might seem a great mess to you, several superscripts and subscripts bind together. And applied examples seem all come from physics, electronic engineering, computer science, it is tremendously hard for average economics students to make any sense out of it. I will make the 'pain' at the lowest level, if you call this a 'pain'. And I have to say, it is for your own interest to know some physics or engineering, since lots of advanced techniques used in economics come from physics and engineering, if you want to know where the these brilliant ideas come from, naturally you will get in touch with physics and engineering.

1.1 Brief History Background

[2]

About two centuries ago, the famous French physicist Joseph Fourier (1768-1830) found that 'every' function[3] could be represented by an infinite trigonometric series resulted from investigation of partial differential equation of heat conduction. Fourier formulated the model describing evolution of the temperature, $T(x,t)$ of a thin wire of length π, in which $\{0 \leq x \leq \pi\}$

[1] You can study my notes *Complex Numbers* first.
[2] Uninterested reader can safely skip this section.
[3] Actually not every function can be.

and t is time. He also assumed zero temperature at the ends: $T(0,t) = 0$ and $T(\pi,t) = 0$. Then

$$f(x) = T(x,0) = \sum_{n=1}^{\infty} b_n \sin nx$$

And he solved b_n by integration,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x$$

He argued that $T(x,t)$ is a solution of heat equation,

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

with boundary condtion,

$$T(0,t) = T(\pi,t) = 0, \quad t \geq 0.$$

Solution can be represented by

$$T(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$$

Of course this naturally becomes the first application of Fourier series.

1.2 Odd or Even

Some properties of function we should review here, since it is a review I won't provide any proof of these theorems, you take them as facts. Actually they are very straightforward, and you can reason them qualitively, needly to waste time on proving.

Theorem 1.1. *Let $f(x)$ be a funtion of which domain is $[-a,a]$, then*

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{-a}^{0} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x \tag{1}$$

if $f(x)$ is an even function. And

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$$

if $f(x)$ is an odd funtion.

Theorem 1.1. *Multiplication and addition rules,*

$$\begin{aligned} odd + odd &= odd \\ odd \times odd &= even \\ even + even &= even \\ even \times even &= even \\ odd \times even &= odd \end{aligned} \tag{2}$$

1.3 Where do we start?

This is a learning notes, it teaches you something you do not know yet, so it would be quite misgiving to illustrate Fourier series with last example.

We start with a simple question, namely, what functions have a representation which can be express by a summation of $\sin kt$ and $\cos kt$,

$$f(t) = \sum_{k=-\infty}^{\infty} c_k[\cos(kt) + i\sin(kt)] \tag{3}$$

i is imaginary number, c_k is a constant. It might looks very strange makes you wondering why on earth would we invent such a linear combination even with imaginary number, the reason is for sake of easy-manipulation. If you are family with complex number, it would immediate remind you **Eular formula**,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Then here we have

$$e^{ikt} = \cos(kt) + i\sin(kt).$$

So the question becomes to ask what fucntions $f(t)$ have a representation,

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \tag{4}$$

In modern Fourier theory it is common to write Fourier series involving complex exponentials. Notice that for every integer k, $e^{ikt} = \cos(kt) +$

$i\sin(kt)$ is a periodic with period 2π. So if $f(t)$ want to be expressed by a trigonometric series as right-hand side, $f(t)$ is necessarily periodic with period 2π.

The second question is if $f(t)$ is periodic 2π and be expressed as trigonometric series, then what are coefficients c_k? Actually this seemingly untractable problem is very easy to handle, recall from your linear algebraic course, that if a inner space product is defined on $[-\pi, \pi]$ as

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t = 0.$$

We say $f(t)$ and $g(t)$ are orthogonal. This *orthogonality condition* has a naturally extension on trigonometry or complex exponential[4],

$$\int_{-\pi}^{\pi} e^{ikt}e^{-ijt}\,\mathrm{d}t = 0 \qquad \text{if } j \neq k$$

We can try to solve this integral,

$$\begin{aligned}\int_{-\pi}^{\pi} e^{ikt}e^{-ijt}\,\mathrm{d}t &= \int_{-\pi}^{\pi} e^{i(k-j)t}\,\mathrm{d}t \\ &= \frac{1}{i(k-j)}e^{i(k-j)t}\bigg|_{-\pi}^{\pi} \\ &= \frac{1}{i(k-j)}\left[e^{i(k-j)\pi} - e^{-i(k-j)\pi}\right] \\ &= 0\end{aligned}$$

The last equation uses the fact that

$$\frac{e^{i(k-j)\pi}}{e^{-i(k-j)\pi}} = e^{i(k-j)2\pi} = \cos[(k-j)2\pi] + i\sin[(k-j)2\pi]$$

According to our basic knowledge of trigonometrics, we know

$$\cos(\alpha 2\pi) = 1$$
$$\sin(\beta 2\pi) = 0.$$

[4] If you have no idea what I am talking about here, please stop for a while and study my notes *Linear Algebra I: Chapter 7 Othogonality* and *A Quick Review on Trigonomtry and Periodicity*.

Then

$$\frac{e^{i(k-j)\pi}}{e^{-i(k-j)\pi}} = 1$$
$$e^{i(k-j)\pi} = e^{-i(k-j)\pi}.$$

In order to use orthogonal condition to eliminate all c_k with $k \neq j$, we multiply e^{-ijt} onto both side of (4) and integrate on interval $[\pi, -\pi]$, yields

$$\int_{-\pi}^{\pi} f(t)e^{-ijt}\,\mathrm{d}t = \sum_{k=1}^{\infty} c_k \int_{-\pi}^{\pi} e^{ikt}e^{-ijt}\,\mathrm{d}t$$

Because of orthogonality condition, all terms on the right-hand side with $k \neq j$are eliminated,

$$\int_{-\pi}^{\pi} f(t)e^{-ijt}\,\mathrm{d}t = c_j \int_{-\pi}^{\pi} e^{ijt}e^{-ijt}\,\mathrm{d}t = c_j \int_{-\pi}^{\pi} \mathrm{d}t = 2\pi c_j$$

Thus,

$$c_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-ijt}\,\mathrm{d}t \tag{5}$$

2 Fourier Series Expansion

For the rest of this notes, we are going to see lots of examples rather than pure theory. We have seen that how we can calculate c_k to expand a function into Fourier series, I will walk you through lots of example in order make you familiar with it.

Before we start, I shall give the most standard form of Fourier series which you can see from textbook,

$$f(t) = a_0 + \sum_{k=-\infty}^{\infty} (a_k \cos kt + b_k \sin kt) \tag{6}$$

This is a bit different from (3), the complex exponential form, keep this in mind, it will be clear where does it come from later on.

2.1 2π Period Expansion

Example 1. *Find the Fourier expansion of*

$$f(t) = \sin{(t + \frac{\pi}{4})}$$

Actually we do not need to compute c_k directly, we can use Euler's formula to manipulate $f(x)$, recall

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

Thus,

$$\begin{aligned} f(t) &= \frac{1}{2i}\left[e^{i(t+\pi/4)} - e^{-i(t+\pi/4)}\right] \\ &= \frac{1}{2i}\left[e^{it}e^{i(\pi/4)} - e^{-it}e^{-i(\pi/4)}\right] \\ &= \frac{1}{2i}\frac{\sqrt{2}}{2}(1+i)e^{it} - \frac{1}{2i}\frac{\sqrt{2}}{2}(1-i)e^{-it} \end{aligned} \tag{7}$$

Because

$$\begin{aligned} e^{i(\pi/4)} &= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}(1+i) \\ e^{-i(\pi/4)} &= \cos{(-\frac{\pi}{4})} + i\sin{(-\frac{\pi}{4})} = \frac{\sqrt{2}}{2}(1-i) \end{aligned}$$

And (7) *is what we called fourier expansion.*

Till now we only derive coefficents in complex exponential form, next we shall follow the standard procedure to derive coefficents in trigonometric form as Fourier himself did. You are not obliged to memorize the procedure, but you will find it is common pattern which is considerably easy to understand.

We integrate

$$f(t) = a_0 + \sum_{k=1}^{\infty}(a_k\cos kt + b_k\sin kt),$$

which is exactly (6), on both sides over interval $[\pi, -\pi]$,

$$\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = a_0\int_{-\pi}^{\pi}\,\mathrm{d}t + \sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos kt\,\mathrm{d}t + b_k\int_{-\pi}^{\pi}\sin kt\,\mathrm{d}t\right)$$

Thus

$$\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = 2\pi a_0$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = a_0$$

Next we need to find a_k, but this time we cannot directly integrate (6), we integrate its modified version,

$$f(t)\cos jt = a_0\cos jt + \sum_{k=1}^{\infty}(a_k\cos kt\cos jt + b_k\sin kt\cos jt).$$

You simply multiply (6) with $\cos jt$ on both sides to get it, for integer $j \geq 1$, then integrate

$$\int_{-\pi}^{\pi} f(t)\cos jt\,\mathrm{d}t = a_0\int_{-\pi}^{\pi}\cos jt\,\mathrm{d}t + \sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos kt\cos jt\,\mathrm{d}t + \sum_{k=1}^{\infty} b_k\int_{-\pi}^{\pi}\sin kt\cos jt\,\mathrm{d}t\right)$$

We can calculate the first integral analytically,

$$\int_{-\pi}^{\pi}\cos jt\,\mathrm{d}t = \frac{1}{j}\sin(jt)\Big|_{-\pi}^{\pi} = 0$$

The second one, use orthogonality condition[5]

$$\int_{-\pi}^{\pi}\cos kt\cos jt\,\mathrm{d}t = \begin{cases} 0 & \text{if } k \neq j \\ \pi & \text{if } k = j \end{cases}$$

If $k = j$, we can prove

$$\begin{aligned}\int_{-\pi}^{\pi}\cos^2 kt\,\mathrm{d}t &= \int_{-\pi}^{\pi}\frac{\cos 2kt + 1}{2}\,\mathrm{d}t \\ &= \frac{1}{2}\int_{-\pi}^{\pi}\cos 2kt\,\mathrm{d}t + \frac{1}{2}\int_{-\pi}^{\pi}\mathrm{d}t \\ &= \frac{1}{2}\left[\frac{1}{2k}\sin 2kt\right]\Bigg|_{-\pi}^{\pi} + \pi \\ &= \pi\end{aligned}$$

[5] Proof is in my notes *Trigonometry*.

Last is also due to orthogonality condition,

$$\int_{-\pi}^{\pi} \cos kt \sin jt \,\mathrm{d}t = 0, \quad \forall k \neq j$$

Then we only left,

$$\int_{-\pi}^{\pi} f(x) \cos kt \,\mathrm{d}t = a_k \pi.$$

Thus

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \,\mathrm{d}t$$

To get b_k is similar with steps above, but multiply both sides of (6) by $\sin jt$, for integer $j \geq 1$.

$$f(t) \sin jt = a_0 \sin jt + \sum_{k=1}^{\infty} (a_k \cos kt \sin jt + b_k \sin kt \sin jt)$$

Integrate t out over $[-\pi, \pi]$,

$$\int_{-\pi}^{\pi} f(t) \sin jt \,\mathrm{d}t = a_0 \int_{-\pi}^{\pi} \sin jt \,\mathrm{d}t + \sum_{k=1}^{\infty} \Bigg(a_k \int_{-\pi}^{\pi} \cos kt \sin jt \,\mathrm{d}t + b_k \int_{-\pi}^{\pi} \sin kt \sin jt \,\mathrm{d}t \Bigg)$$

The first integration term is

$$a_0 \int_{-\pi}^{\pi} \sin jt \,\mathrm{d}t = a_0 \left[-\frac{1}{j} \cos jt \right] \Bigg|_{-\pi}^{\pi} = 0$$

The second, follow orthogonality condition,

$$a_k \int_{-\pi}^{\pi} \cos kt \sin jt \,\mathrm{d}t = 0, \forall \quad k, j.$$

The third,

$$b_k \int_{-\pi}^{\pi} \sin kt \sin jt \,\mathrm{d}t = \begin{cases} 0 & \text{if } k \neq j \\ \pi & \text{if} k = j \end{cases}$$

Because if $k = j$, then we have

$$\begin{aligned}
b_k \int_{-\pi}^{\pi} \sin^2 kt \,\mathrm{d}t &= b_k \int_{-\pi}^{\pi} \left(\frac{1-\cos 2kt}{2}\right) \mathrm{d}t \\
&= b_k \left(\frac{1}{2}\int_{-\pi}^{\pi} \mathrm{d}t - \frac{1}{2}\int_{-\pi}^{\pi} \cos 2kt \,\mathrm{d}t\right) \\
&= b_k\pi - b_k \frac{1}{2}\left[\frac{1}{kt}\sin 2kt\right]\Bigg|_{-\pi}^{\pi} \\
&= b_k\pi
\end{aligned}$$

Then use all above result, we only left,

$$\int_{-\pi}^{\pi} f(t) \sin kt = b_k\pi$$

Thus,

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin kt \,\mathrm{d}t$$

In together, our Fourier series are,

$$f(t) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt) \qquad \text{(Fourier 1)}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \,\mathrm{d}t \qquad \text{(Fourier 2)}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos kt \,\mathrm{d}t \qquad \text{(Fourier 3)}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin kt \,\mathrm{d}t \qquad \text{(Fourier 4)}$$

Next we shall use a full example to walk you through a process of Fourier expansion of period 2π.

Example 1. *Find the Fourier series of*

$$f(t) = \left|\sin \frac{x}{2}\right|$$

Use formula (Fourier 2), we get

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sin \frac{t}{2}\right| \mathrm{d}t$$

Reserve a picture for $\left|\sin\frac{x}{2}\right|$

Figure 1: $\left|\sin\frac{x}{2}\right|$

From graph, we know it is even function and $\sin t/2 \geq 0$ over interval $[0, \pi]$, then we can remove absolute value operator, and use half interval,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{\pi} \sin\frac{t}{2}\,\mathrm{d}t \\ &= \frac{1}{\pi}\left[-2\cos\frac{t}{2}\right]\Big|_0^{\pi} \\ &= \frac{1}{\pi}\left(-2\cos\frac{\pi}{2} + 2\cos\frac{0}{2}\right) \\ &= \frac{2}{\pi} \end{aligned}$$

Use formula (Fourier 3), we get

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \left|\sin\frac{t}{2}\right| \cos kt\,\mathrm{d}t$$

Even function multiply even funtion is an even function,

$$a_k = \frac{2}{\pi}\int_0^{\pi} \sin\frac{t}{2}\cos kt\,\mathrm{d}t$$

We can't use orthogonality condition here, because k is an integer and $k \geq 1$. So use trigonometric product formula, yields

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_0^{\pi}\left[\sin\left(\frac{t}{2}+kt\right)+\sin\left(\frac{t}{2}-kt\right)\right]\mathrm{d}t \\ &= \frac{1}{\pi}\int_0^{\pi}\left[\sin\left(\frac{1+2k}{2}t\right)+\sin\left(\frac{1-2k}{2}t\right)\right]\mathrm{d}t \\ &= \frac{1}{\pi}\left[-\frac{2}{1+2k}\cos\frac{1+2k}{2}t-\frac{2}{1-2k}\cos\frac{1-2k}{2}t\right]\Big|_0^{\pi} \\ &= \frac{1}{\pi}\left[\frac{4}{1+2k}+\frac{4}{1-2k}\right] \\ &= \frac{1}{\pi}\left(\frac{8}{1-4k^2}\right) \end{aligned}$$

Similarly, b_k

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \left|\sin\frac{x}{2}\right| \sin kt = 0$$

This is not because of orthogonality, look at integral carfully, recall your high school math again, it is an even function multiply an odd function, which results an odd function. Besides the interval is symmetric on origin, the area on both sides cancel out.

So here is the final result of Fourier expansion,

$$\left|\sin\frac{x}{2}\right| = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{1}{\pi}\left(\frac{8}{1-4k^2}\right)\cos kt$$

We can control how close the trigonometric series can approximate function by choosing n,

$$S_n(t) = \left|\sin\frac{x}{2}\right| = \frac{2}{\pi} + \sum_{k=1}^{n} \frac{1}{\pi}\left(\frac{8}{1-4k^2}\right)\cos kt$$

Reserve a picture for Series approximation

Figure 2: Series approximation

2.2 Arbitrary Period Expansion

Above we mainly assume that period is 2π, actually we can modify Fourier series to apply to functions have period $2l$, of which $l \neq \pi$. We can set $\tau = \frac{\pi}{l}t$, but we do not work on τ directly,

$$f(t) = \sum_{k=1}^{\infty} c_k e^{ik\tau} = \sum_{k=1}^{\infty} c_k e^{ik\frac{\pi}{l}t} \tag{8}$$

and

$$c_k = \frac{1}{2l}\int_{-l}^{l} f(t)e^{-ik\tau} = \frac{1}{2l}\int_{-l}^{l} f(t)e^{-ik\frac{\pi}{l}t} \tag{9}$$

We can check the periodicity,

$$e^{ik\frac{\pi}{l}(t+2l)} = e^{ik\frac{\pi}{l}t}e^{i2k\pi} = e^{ik\frac{\pi}{l}t},$$

by using the fact that

$$e^{i2k\pi} = \cos{(2k\pi)} + i\sin{(2k\pi)} = 1.$$

Deriving c_k directly following formula,

$$c_k = \frac{1}{2l}\int_{-l}^{l} f(t)e^{ik\frac{\pi}{l}t}\,\mathrm{d}t$$

$$c_k = \frac{1}{2l}\int_{-l}^{l} e^{-im\frac{\pi}{l}t}e^{ik\frac{\pi}{l}t}\,\mathrm{d}t = \begin{cases} 2l & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}$$

Continue our investigation furthur, we now try to turn (8) into sine and cosine form, use Euler's formula again, the symmetry here will be made use of,

$$e^{ik\frac{\pi}{l}t} = \cos\frac{k\pi t}{l} + i\sin\frac{k\pi t}{l},$$

substitute into (8), and we make a slight change on lower limit into negative infinity,

$$f(t) = \sum_{k=-\infty}^{\infty} c_k\bigg(\cos\frac{k\pi t}{l} + i\sin\frac{k\pi t}{l}\bigg)$$

First, check the situation when $k = 0$,

$$c_0\bigg(\cos\frac{0\pi t}{l} + i\sin\frac{0\pi t}{l}\bigg) = c_0$$

Second, check when $k = 1$ and $k = -1$ together,

$$\begin{aligned} & c_1\bigg(\cos\frac{1\pi t}{l} + i\sin\frac{1\pi t}{l}\bigg) + c_{-1}\bigg(\cos\frac{-1\pi t}{l} + i\sin\frac{-1\pi t}{l}\bigg) \\ &= c_1\bigg(\cos\frac{\pi t}{l} + i\sin\frac{\pi t}{l}\bigg) + c_{-1}\bigg(\cos\frac{\pi t}{l} - i\sin\frac{\pi t}{l}\bigg) \\ &= \cos\frac{\pi t}{l}(c_1 + c_{-1}) + \sin\frac{\pi t}{l}(ic_1 - ic_{-1}) \end{aligned}$$

Again, check when $k = 2$ and $k = -2$ together,

$$\begin{aligned} & c_2\bigg(\cos\frac{2\pi t}{l} + i\sin\frac{2\pi t}{l}\bigg) + c_{-2}\bigg(\cos\frac{-2\pi t}{l} + i\sin\frac{-2\pi t}{l}\bigg) \\ &= \cos\frac{\pi t}{l}(c_2 + c_{-2}) + \sin\frac{\pi t}{l}(ic_2 - ic_{-2}) \end{aligned}$$

As you can guess, $k = 3$ and $k = -3$,

$$\cos\frac{\pi t}{l}(c_3 + c_{-3}) + \sin\frac{\pi t}{l}(ic_3 - ic_{-3})$$

Keep on doing this, and add them up becomes a trigonometric series,

$$\begin{aligned} f(t) = c_0 &+ \cos\frac{\pi t}{l}(c_1 + c_{-1}) + \sin\frac{\pi t}{l}(ic_1 - ic_{-1}) \\ &+ \cos\frac{\pi t}{l}(c_2 + c_{-2}) + \sin\frac{\pi t}{l}(ic_2 - ic_{-2}) \\ &+ \cos\frac{\pi t}{l}(c_3 + c_{-3}) + \sin\frac{\pi t}{l}(ic_3 - ic_{-3}) \dots \end{aligned}$$

c_0 will be newly notated as $\frac{a_0}{2}$, $c_1 + c_{-1}$ renamed to a_1, $c_1 + c_{-1}$ renamed to a_2, etc. $ic_1 - ic_{-1}$ to b_1, $ic_2 - ic_{-2}$ to b_2. Then we have

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi t}{l} + b_k \sin\frac{k\pi t}{l}\right)$$

Use (9),

$$\begin{aligned} a_k = c_k + c_{-k} &= \frac{1}{2l}\int_{-l}^{l} f(t)e^{-ik\frac{\pi}{l}t}\,\mathrm{d}t + \frac{1}{2l}\int_{-l}^{l} f(t)e^{ik\frac{\pi}{l}t}\,\mathrm{d}t \\ &= \frac{1}{2l}\int_{-l}^{l}\left[f(t)e^{-ik\frac{\pi}{l}t} + f(t)e^{ik\frac{\pi}{l}t}\right]\mathrm{d}t \\ &= \frac{1}{2l}\int_{-l}^{l}\left[f(t)\left(e^{-ik\frac{\pi}{l}t} + e^{ik\frac{\pi}{l}t}\right)\right]\mathrm{d}t \\ &= \frac{1}{l}\int_{-l}^{l} f(t)\cos\left(k\frac{\pi}{l}t\right)\mathrm{d}t \end{aligned}$$

Last equation uses the fact that,

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}).$$

And,

$$
\begin{aligned}
b_k = i(c_k - c_{-k}) &= \frac{1}{2l}\int_{-l}^{l} f(t) i e^{-ik\frac{\pi}{l}t}\,\mathrm{d}t - \frac{1}{2l}\int_{-l}^{l} f(t) i e^{ik\frac{\pi}{l}t}\,\mathrm{d}t \\
&= \frac{1}{2l}\int_{-l}^{l}\left[f(t) i e^{-ik\frac{\pi}{l}t} - f(t) i e^{ik\frac{\pi}{l}t}\right]\mathrm{d}t \\
&= \frac{i}{l}\frac{1}{2i}\int_{-l}^{l}\left[f(t) i\left(e^{-ik\frac{\pi}{l}t} - e^{ik\frac{\pi}{l}t}\right)\right]\mathrm{d}t \\
&= \frac{1}{l}\frac{1}{2i}\int_{-l}^{l}\left[f(t)\left(e^{ik\frac{\pi}{l}t} - e^{-ik\frac{\pi}{l}t}\right)\right]\mathrm{d}t \\
&= \frac{1}{l}\int_{-l}^{l} f(t)\sin\left(k\frac{\pi}{l}t\right)\mathrm{d}t
\end{aligned}
$$

Last equation uses the fact that,

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

3 Fourier Transform

Here we are, finally to the hardcore. All knowledges I presented before are just tools we are about employ here. Basically, the idea of Fourier series is remarkably easy, you have a $f(x)$ which is period funtion, you substitute it into formulas to calculate either for c_k in complex exponential version or for a_0, a_k and b_k in trigonometric version. And they are perfectly interchangably, nothing mysterious, you need more practice to be familiar with them. But since we never stop propagating that Fourier analysis is an extremely powerful weapon in your mathematics arsonal, it should be extensively made use of rather than dealing with periodic function alone. Right, of course we have a knack for it.

3.1 Periodic Extension

As a matter of fact, I would not even want to call this a trick, perferably it is new perspective how you view functions. The figure shows function $f(x)$ with domain $(0, l)$, if we take l as its period, we can extend throughout the

Reserve a picture for period extension1

Figure 3: period extension1

Reserve a picture for period extension2

Figure 4: period extension2

whole axis, as the second graph shows. Unfortuantly this period extension is useless, or most of period extensions are useless, because we only interested in interval $(0, l)$, who do we even bother to extend periods over the whole axis, that changes nothing at all.

There are two useful period extension we need to study, the first one is **odd period extension**, the second is **even period extension**.

Given any $f(x)$ with domain $(0, l)$, odd period extension is defined by conditions,

1. $F^o(x) = f(x)$ for $0 < x < l$
2. $F^o(x) = -f(x)$ for $-l < x < 0$
3. $F^o(x)$ has period $2l$

And even period extension is defined by conditions,

1. $F^e = f(x)$ for $0 < x < l$
2. $F^e = f(x)$ for $-l < x < 0$
3. F^e has period $2l$

Now recall and ask yourself, what kind of function would it be if you plus an odd function with an even one? Neither. Check theorem 1.2. I did not say anything about addition of odd and even fuctions, because result is neither of them. Take a look at trigonometric version of Fourier series again,

$$f(x) = a_0 + \sum_{k=0}^{\infty} (a_k \cos kt + b_k \sin kt)$$

Surprised, right? If I did not mention this you won't notice that Fourier series is a non-odd-or-even function, then F^e has a cosine expansion and F^o has a sine expansion, it has to be like this, because we are unable to express an even(odd) funtion with a function which is summation of addition of odd and even functions.

Although it is now a periodic function throughout the whole axis, we merely interested in interval $(0, l)$. We perform Fourier expansion on this interval,

$$\begin{aligned} f(x) = F^e(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{l}\right) \\ = F^o(x) &= \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{l}\right) \end{aligned}$$

Notice that we use $\frac{a_0}{2}$ rather than a_0, because it is more mathematically tractable, we can write $\frac{a_0'}{2} = a_0$ to differentiate from a_0 before, but we do not need a_0 any more from now on, why don't we just take $\frac{a_0}{2}$ as our new a_0. Then follow the formula Fourier 2, Fourier 3, and Fourier 4,

$$\begin{aligned} \frac{a_0}{2} &= \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x = \frac{1}{l}\int_0^l F^e\,\mathrm{d}x \\ a_k &= \frac{2}{l}\int_0^l f(x)\cos\left(\frac{k\pi x}{l}\right)\mathrm{d}x = \frac{2}{l}\int_0^l F^e(x)\cos\left(\frac{k\pi x}{l}\right)\mathrm{d}x \\ b_k &= \frac{2}{l}\int_0^l f(x)\sin\left(\frac{k\pi x}{l}\right)\mathrm{d}x = \frac{2}{l}\int_0^l F^o(x)\sin\left(\frac{k\pi x}{l}\right)\mathrm{d}x \end{aligned}$$

Notice we make use of property of even function of integral 1.1, both $F^e(x)\cos\left(\frac{k\pi x}{l}\right)$ and $F^o(x)\sin\left(\frac{k\pi x}{l}\right)$ are even functions due to the facts of multiplication and addition properties 2. So the basic idea here it to make use of both odd period and even period extensions use calculate its Fourier expansion.

3.1.1 A Simple Example

We choose a simplest example here, $f(x)$ is defined over $0 < x < \pi$, $f(x) = 1$ for all $0 < x < \pi$. Its even period expansion and odd period expansion graphs are shown below,

Reserve a picture for Odd $f(x)$

Figure 5: Odd $f(x)$

Reserve a picture for Even $f(x)$

Figure 6: Even $f(x)$

Both have period 2π, so

$$F^o(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{l}\right) = \sum_{k=1}^{\infty} b_k \sin kx,$$

compute b_k,

$$\begin{aligned} b_k &= \frac{2}{l}\int_0^l F^o(x)\sin\left(\frac{k\pi x}{l}\right)\mathrm{d}x = \frac{2}{l}\int_0^l 1\sin\left(\frac{k\pi x}{l}\right)\mathrm{d}x \\ &= \frac{2}{\pi}\Big[-\frac{1}{k}\cos(kx)\Big]\Big|_0^\pi \\ &= \frac{2}{\pi}\Big[-\frac{1}{k}\cos(k\pi) + \frac{1}{k}\cos 0\Big] \end{aligned}$$

if k is odd,

$$\begin{aligned} &= \frac{2}{\pi}\left(\frac{1}{k} + \frac{1}{k}\right) \\ &= \frac{2}{\pi}\frac{2}{k} \\ &= \frac{4}{k\pi} \end{aligned}$$

or if k is even

$$\begin{aligned} &= \frac{2}{\pi}\left(-\frac{1}{k} + \frac{1}{k}\right) \\ &= 0 \end{aligned}$$

Odd period expansion is

$$F^o(x) = \sum_{k=1+2t}^{\infty} \frac{4}{k\pi} \sin{(kx)} \qquad t \text{ is positive integer starts from } 1$$

Turn to even expansion,

$$F^e(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos{(kx)}$$

compute a_k,

$$\begin{aligned}
a_k &= \frac{2}{\pi} \int_0^{\pi} F^e(x) \cos{(kx)} \,\mathrm{d}x = \frac{2}{\pi} \int_0^{\pi} 1 \cos{(kx)} \,\mathrm{d}x \\
&= \frac{2}{\pi} \left[\frac{1}{k} \sin{(kx)} \right] \Bigg|_0^{\pi} \\
&= \frac{2}{\pi} \left[\frac{1}{k} \sin{(k\pi)} - \frac{1}{k} \sin 0 \right] \\
&= 0
\end{aligned}$$

And we also need to know $\frac{a_0}{2}$,

$$\frac{a_0}{2} = \frac{1}{\pi} \int_0^{\pi} \,\mathrm{d}x = \frac{1}{\pi} [x]|_0^{\pi} = 1$$

Then even expansion is

$$F^e(x) = \frac{a_0}{2} = 1$$

3.2 Fourier Transform

From now on, we will stick to exponential verison of Fourier series which will save us considerable trouble in writing and computation. Recall the most important thing, Fourier series of period $2l$, we have learned by far,

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{l}t} \qquad c_k = \frac{1}{2l} \int_{-l}^{l} f(t) e^{-ik\frac{\pi}{l}t} \,\mathrm{d}t$$

From now on, we no longer impose the restriction of period functions, you will see later that Fourier analysis deals with function from a broader range than what we will be doing here.

For heuristic reasons, we will use a 'special' function here,

$$f(t) = 0, \qquad |t| > L, \ L \text{ is any arbitrary positve number}$$

It might be strange why we want a function looks like this, it will be clear soon. We can get a Fourier expansion for part of $f(t)$ with $-L < t < L$ by using periodic extension. Define $F_L(t) = f(t)$ for $-L < t \leq L$ and $F_L(t)$ has period of $2L$. Naturally, we have

$$F_L = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{L}t} \qquad c_k = \frac{1}{2L}\int_{-L}^{L} f(t)e^{-ik\frac{\pi}{L}t}\,\mathrm{d}t \tag{10}$$

Essentially, we have not changed anything critical. The idea is to make the original function a segment (a full period) of whole periodic funtion, then we can use Fourier tools to analyse it.

If we define the k^{th} frequency to be $\omega_k = k\frac{\pi}{L}$, then calculate the first difference,

$$\omega_{k+1} - \omega_k = (k+1)\frac{\pi}{L} - k\frac{\pi}{L} = \frac{\pi}{L} = \Delta\omega \tag{11}$$

Conventionally, we also define

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t 6} \tag{12}$$

Look at (10), what are the differences between the equation above and (10)? If you have done your calculus well in your first year math, you would immediately tell that,

$$c_k = \frac{1}{2L}\int_{-L}^{L} f(t)e^{-ik\frac{\pi}{L}t}\,\mathrm{d}t = \frac{1}{2L}\int_{-\infty}^{\infty} f(t)e^{-ik\frac{\pi}{L}t}\,\mathrm{d}t$$

Because $f(t) = 0$ for all $|t| \geq L$, then there is no difference whether we set interval as $[-L, L]$ or $[-\infty, \infty]$. Next, use $\omega_k = k\frac{\pi}{L}$, yields

$$c_k = \frac{1}{2L}\int_{-\infty}^{\infty} f(t)e^{-i\omega_k t}\,\mathrm{d}t. = \frac{1}{2L}\hat{f}(\omega_k)$$

[6] This is actually Fourier transform, but it is not clear why it looks like this at the moment.

This isn't over yet, try not to be lost here. We want to put $\Delta\omega$ into last equation by using $L = \frac{\pi}{\Delta\omega}$ from (11),

$$c_k = \frac{1}{2L}\hat{f}(\omega_k) = \frac{1}{2\pi}\hat{f}(\omega_k)\Delta\omega$$

What we are doing above is trying to prepare element(such as $\Delta\omega$) for an integral. So the question remains 'what integral is it?', then watch carefully below.

Substitute $c_k = \frac{1}{2\pi}\hat{f}(\omega)\Delta\omega$ back to F_L in (10),

$$f(t) = F_L(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi}\hat{f}(\omega_k)\Delta\omega e^{ik\frac{\pi}{L}t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \hat{f}(\omega)e^{i\omega_k t}\Delta\omega$$

As your basic integral knowledges tell you,

$$\frac{1}{2\pi}\lim_{\Delta\omega\to 0}\sum_{k=-\infty}^{\infty} \hat{f}(\omega)e^{i\omega_k t}\Delta\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,\mathrm{d}\omega \tag{13}$$

A notice has to be made here, we are actually letting $L \to \infty$, because from (11) $\frac{\pi}{L} = \Delta\omega$, results $\Delta\omega \to 0$.

Together with (12) and (13), we have Fourier transform and its inverse Fourier transform,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,\mathrm{d}\omega \tag{14}$$

These are Fourier transform pairs, conventionally we denote $\hat{f}(\omega)$ as the Fourier transform of $f(t)$. A function $f(t)$ can go back and forth from its time domain to frequency domain by Fourier transform and its inverse operation, only if the function is continuous and integrable. Some textbook use $\mathcal{F}[f(t)] = \hat{f}(\omega)$ as Fourier transform operator, which we seldom need it.

3.3 Properties of the Fourier Transform

This is the most important part of Fourier theory, but I still have to say, nothing is really difficult here, you just need to get familiar with these notations and operations. We will present some examples to further your understanding.

3.3.1 Linearity

Let α and β be any constants, $f(t)$ and $g(t)$ two continuous functions, and a linear combination $h(t) = \alpha f(t) + \beta g(t)$, the Fourier transform of $h(t)$ is

$$\begin{aligned}
\hat{h}(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} [\alpha f(t) + \beta g(t)]e^{-i\omega t}\,\mathrm{d}t \\
&= \alpha\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t + \beta\int_{-\infty}^{\infty} g(t)e^{-i\omega t}\,\mathrm{d}t \\
&= \alpha\hat{f}(\omega) + \beta\hat{g}(\omega)
\end{aligned}$$

Simply speaking, linearity of Fourier transform is completely due to the linearity of integration rules.

3.3.2 Time Shifting

Reserve a picture for time shifting

Figure 7: Time shifting

Suppose we have a function $h(t) = f(t - t_0)$, if $t_0 > 0$, use your high school math, it is clear that $h(t)$ is to the right of $f(t)$ as figure shows.

Fourier transform of $h(t)$ is

$$\begin{aligned}
\hat{h}(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(t - t_0)e^{-i\omega t}\,\mathrm{d}t
\end{aligned}$$

If we denote $t' = t - t_0$,

$$= \int_{-\infty}^{\infty} f(t')e^{-i\omega(t'+t_0)}\,\mathrm{d}t$$

Also we calculate the differential, $\mathrm{d}t = \mathrm{d}(t' - t_0) = \mathrm{d}t'$,

$$\begin{aligned}
&= \int_{-\infty}^{\infty} f(t')e^{-i\omega(t'+t_0)}\,\mathrm{d}t' \\
&= \int_{-\infty}^{\infty} f(t')e^{-i\omega t'}e^{-i\omega t_0}\,\mathrm{d}t'
\end{aligned}$$

$e^{-i\omega t_0}$ is constant,

$$
\begin{aligned}
&= e^{-i\omega t_0} \int_{-\infty}^{\infty} f(t') e^{-i\omega t'} \, \mathrm{d}t' \\
&= e^{-i\omega t_0} \hat{f}(\omega)
\end{aligned}
$$

The last equation holds because Fourier transform changes the function regardless of any argument.

3.3.3 Scaling

If we build a new funtion,

$$h(t) = f\left(\frac{t}{\alpha}\right), \text{ by a scaling factor } \alpha > 0.$$

Fourier transform is

$$
\begin{aligned}
\hat{h}(\omega) &= \int_{-\infty}^{\infty} h(t) e^{-i\omega t} \, \mathrm{d}t \\
&= \int_{-\infty}^{\infty} f\left(\frac{t}{\alpha}\right) e^{-i\omega t} \, \mathrm{d}t
\end{aligned}
$$

We denote $t = \alpha t'$

$$= \int_{-\infty}^{\infty} f(t') e^{-i\omega\alpha t'} \, \mathrm{d}t$$

We calculate the differential, $\mathrm{d}t = \mathrm{d}(\alpha t') = \alpha \mathrm{d}t'$

$$
\begin{aligned}
&= \alpha \int_{-\infty}^{\infty} f(t') e^{-i\omega\alpha t'} \, \mathrm{d}t' \\
&= \alpha \hat{f}(\alpha\omega)
\end{aligned}
$$

The last equation holds because we view $\alpha\omega$ as the new frequency (don't forget that α is a scaling factor).

3.3.4 Differentiation

Let $h(t) = f'(t)$, then Fourier transform is

$$\begin{aligned}\hat{h}(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} f'(t)e^{-i\omega t}\,\mathrm{d}t\end{aligned}$$

Integrate by parts, $u = e^{-i\omega t}$, so $\mathrm{d}u = -i\omega e^{-i\omega t}\mathrm{d}t$. Also $v = f(t)$, then $\mathrm{d}v = f'(t)\mathrm{d}t$.

$$\begin{aligned}&= \int_{-\infty}^{\infty} u\mathrm{d}v \\ &= uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\mathrm{d}u \\ &= e^{-i\omega t}f(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(-i\omega)e^{-i\omega t}\mathrm{d}t\end{aligned}$$

Assume that $f(\pm\infty) = 0$

$$\begin{aligned}&= -\int_{-\infty}^{\infty} f(t)(-i\omega)e^{-i\omega t}\mathrm{d}t \\ &= i\omega\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t \\ &= i\omega\hat{f}(\omega)\end{aligned}$$

3.3.5 Parseval's Relation

We define the energy of signal $f(t)$ is

$$\int_{-\infty}^{\infty} |f(t)|^2\mathrm{d}t = \int_{-\infty}^{\infty} f(t)\overline{f(t)}\mathrm{d}t$$

$\overline{f(t)}$ is the conjugate of $f(t)$,

$$\overline{f(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)e^{i\omega t}}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)}e^{-i\omega t}\mathrm{d}\omega$$

Substitute back into last equation,

$$\int_{-\infty}^{\infty} f(t)\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)}e^{-i\omega t}\mathrm{d}\omega\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\overline{\hat{f}(\omega)}e^{-i\omega t}\mathrm{d}\omega\mathrm{d}t$$

Iterative integration,

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)}\left[\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t\right]\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)}\hat{f}(\omega)\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\mathrm{d}\omega
\end{aligned}$$

What we get is the formula of **Parseval's relation**,

$$\int_{-\infty}^{\infty} |f(t)|^2\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\mathrm{d}\omega$$

$|\hat{f}(\omega)|^2$ is called **power spectrum** of $f(t)$, then $|\hat{f}(\omega)|$ is **Fourier spectrum**.

3.3.6 Duality

If we have a $\hat{f}(\omega)$, we exchange the variable of time and frequency, and set $\hat{f}(t) = g(t)$. The Fourier transform of $g(t)$ is,

$$\begin{aligned}
\hat{g}(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \hat{f}(t)e^{-i\omega t}\,\mathrm{d}t
\end{aligned}$$

Change notation, $t = s$

$$= \int_{-\infty}^{\infty} \hat{f}(s)e^{-i\omega s}\,\mathrm{d}s$$

Inverse Fourier transform of $f(t)$ is,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,\mathrm{d}\omega$$

Change notation, $\omega = s$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)e^{ist}\,\mathrm{d}s$$

Together with

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} \hat{f}(s) e^{-i\omega s}\,\mathrm{d}s$$

We find that,

$$\hat{g}(\omega) = 2\pi f(-\omega)$$

3.3.7 Convolution

A *filter* is described by a funtion of $\hat{H}(\omega)$, for instance, $\hat{H}(\omega) = 1$ for desirable frequency and $\hat{H}(\omega) = 0$ for undesirable frequency. If a signal $f(t)$ is fed into the filter, an output is $g(t)$, its Fourier transform, we set as,

$$\hat{f}(\omega)\hat{H}(\omega) = \hat{g}(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t}\,\mathrm{d}t$$

Inverse Fourier transform,

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{H}(\omega) e^{i\omega t}\,\mathrm{d}\omega$$

$\hat{f}(\omega)$ is easy to get, it is just a Fourier transform of $f(t)$, but we need to change notation,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau}\,\mathrm{d}\tau$$

And substitute into $g(t)$,

$$\begin{aligned} g(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau}\,\mathrm{d}\tau \hat{H}(\omega) e^{i\omega t}\,\mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} f(\tau) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{H}(\omega) e^{i\omega(t-\tau)}\,\mathrm{d}\omega \right] \mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} f(\tau) H(t-\tau)\,\mathrm{d}\tau \end{aligned}$$

The last equation is called **convolution**, formally denoted

$$(f * H)(t) = \int_{-\infty}^{\infty} f(\tau) H(t-\tau)\,\mathrm{d}\tau \tag{15}$$

The above relationship can be summarized,

$$\hat{g}(\omega) = \hat{f}(\omega)\hat{H}(\omega)$$

$$g(t) = (f * H)(t)$$

3.3.8 Impulses

$H(t)$ of $\hat{H}(\omega)$ is call the **impulse response function** of the filter, because it is the output generated when the input is an impulse at time 0. $\delta(t)$[7] denotes impulse funtion, takes the value of 0 for all $t \neq 0$ and value $+\infty$ at time $t = 0$. Because it is so infinite, the area under graph is 1. One

Reserve a picture for impulse

Figure 8: Impulse response

generalization as graph shows,

Reserve a picture for impulse generalization

Figure 9: Impulse generalization

$$\delta_\varepsilon(t) = \begin{cases} \dfrac{1}{2\varepsilon} & \text{if } -\varepsilon < t < \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

If $\varepsilon \to \infty$, the function above approximate delta function. Unfinished...

3.4 Examples

After studying so many properties[8], we really to study through some examples to make sure we comprehend them.

Example 1 We have a signal turned on at $t = 0$ then decays exponentially,

$$f(t) = \begin{cases} e^{-at} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

[7] It is called Dirac delta function.

[8] Actually these are only a portion of its all properties.

from some $a > 0$. The Fourier transform of this signal is

$$
\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t = \int_{-\infty}^{\infty} e^{-at}e^{-i\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} e^{-t(a+i\omega)}\,\mathrm{d}t \\
&= \left.\frac{e^{-t(a+i\omega)}}{-(a+i\omega)}\right|_{0}^{\infty} \\
&= -\frac{e^{-0(a+i\omega)}}{-(a+i\omega)} \\
&= \frac{1}{a+i\omega}
\end{aligned}
$$

Example 2 We have boxcar normalized function as below, signal turned

Reserve a picture for impulse generalization

Figure 10: Impulse generalization

on at $t = -\frac{1}{2}$ and off at $t = \frac{1}{2}$, impulse height is 1. Denote

$$
\mathrm{rect}(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}
$$

'rect' means rectangle. The Fourier transform of the signal is

$$
\begin{aligned}
\widehat{\mathrm{rect}}(\omega) &= \int_{-\infty}^{\infty} \mathrm{rect}(t)e^{-i\omega t}\,\mathrm{d}t \\
&= \int_{-1/2}^{1/2} e^{-i\omega t}\,\mathrm{d}t \\
&= \left.\frac{e^{-i\omega t}}{-i\omega}\right|_{-1/2}^{1/2} \\
&= \frac{e^{-i\omega/2}}{-i\omega} - \frac{e^{i\omega/2}}{-i\omega} \\
&= \frac{e^{i\omega/2} - e^{-i\omega/2}}{i\omega}
\end{aligned}
$$

We can stop here, or we use Euler's formula,

$$
\begin{aligned}
&= \frac{1}{i\omega}\left(\cos\frac{\omega}{2} + i\sin\frac{\omega}{2} - \cos\frac{\omega}{2} + i\sin\frac{\omega}{2}\right) \\
&= \frac{1}{i\omega} 2i \sin\frac{\omega}{2} \\
&= \frac{2}{\omega}\sin\frac{\omega}{2}
\end{aligned}
$$

There is a function named 'sinc', $\operatorname{sinc}\omega = \frac{\sin\omega}{\omega}$. Then $\widehat{\text{rect}}(\omega) = \operatorname{sinc}\frac{\omega}{2}$.

Example 3 We have a rectangular pulse of height H, width W and centre C.

Reserve a picture for rectangular impulse

Figure 11: rectangular impulse

We can denote this rectangular impulse function,

$$r(t) = H\text{rect}\left(\frac{t-C}{W}\right)$$

from previous knowledge of shifting and scaling, we shift *rect*[9] by C to the right, and scale by a factor of W. The signal gets turned on at $C - \frac{W}{2}$ and off at $C + \frac{W}{2}$. This example is just a generalization of last one, study it carefully, it gives a general method to cope with rectagular impulse functions. We build the method in three steps.

The *first step*, we consider the scaling alone, assuming no shifting $C = 0$. Then we denote

$$f(t) = \text{rect}(t), \qquad h(t) = r_1(t) = \text{rect}\left(\frac{t}{W}\right) = f\left(\frac{t}{W}\right)$$

Its Fourier transform according to scaling property is,

$$\hat{h}(\omega) = \hat{r}_1(\omega) = W\widehat{\text{rect}}(W\omega) \tag{16}$$

[9] rect is always a normalized boxcar impulse function without other explicit emphasizing.

Use sinc function,

$$\widehat{\text{rect}}(W\omega) = \text{sinc}\,\frac{W\omega}{2} = \frac{\sin\frac{W\omega}{2}}{\frac{W\omega}{2}} = \frac{2}{W\omega}\sin\frac{W\omega}{2}$$

Thus,

$$\hat{h}(\omega) = \hat{r}_1(\omega) = \frac{2}{\omega}\sin\frac{W\omega}{2}$$

The *second step*, we allow a nonzero C, denote

$$f(t) = r_1(h), \qquad h(t) = r_2(t) = \text{rect}\left(\frac{t-C}{W}\right) = r_1(t-C)$$

According to property of time shifting, its Fourier transform is

$$\hat{h}(\omega) = \hat{r}_2(\omega) = e^{-i\omega C}\hat{f}(\omega) = e^{-i\omega C}\hat{r}_1(\omega)$$

Use equation (16),

$$\hat{h}(\omega) = \hat{r}_2(\omega) = e^{-i\omega C}W\widehat{\text{rect}}(W\omega)$$

The *last step*, use linearity property, we denote

$$\hat{r}(\omega) = H\hat{r}_2(\omega) = He^{-i\omega C}W\widehat{\text{rect}}(W\omega)$$

Use sinc function again,

$$\begin{aligned} He^{-i\omega C}W\widehat{\text{rect}}(W\omega) &= He^{-i\omega C}W\,\text{sinc}\,\frac{W\omega}{2} \\ &= He^{-i\omega C}W\frac{\sin\frac{W\omega}{2}}{\frac{W\omega}{2}} \\ \hat{r}(\omega) &= \frac{2H}{\omega}e^{-i\omega C}\sin\frac{W\omega}{2} \end{aligned} \tag{17}$$

Notation might look a little bit messy, but the idea is very simple that we don't need to perform a Fourier transform directly on rectangular impulse function, we can decompose the steps into three, each step uses the result of last step's Fourier transform. Don't get lost at notation, focus on the essence.

Reserve a picture for several rectangular impulses

Figure 12: several rectangular impulses

Example 4 Here is another generalized example 12 of rectangular impulse functions,

$$s_n(t) = r(t) \text{ with } \begin{cases} H = 2, \quad W = 1, \quad C = -1.5 & \text{for n=1} \\ H = 1, \quad W = 2, \quad C = 1 & \text{for n=2} \\ H = 0.5, \quad W = 2, \quad C = 3 & \text{for n=3} \end{cases}$$

We define $s(t) = s_1(t) + s_2(t) + s_3(t)$, use result from last example (17),

$$s(t) = \frac{4}{\omega} e^{\frac{3}{2} i\omega} \sin \frac{\omega}{2} + \frac{2}{\omega} e^{-i\omega} \sin \omega + \frac{1}{\omega} e^{-3i\omega} \sin \omega$$

Example 5 Suppose we use filter

$$\hat{H}(\omega) = \frac{2}{\omega} \sin \frac{\omega}{2}.$$

To see it in time domain,

$$H(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Apply the filter to a signal $f(t)$, output at time t is,

$$(f * H)(t) = \int_{-\infty}^{\infty} f(t - \tau) H(\tau) \, \mathrm{d}\tau = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t - \tau) \, \mathrm{d}\tau$$

Define $\tau' = t - \tau$, then $\mathrm{d}\tau' = -\mathrm{d}\tau$, thus

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t - \tau) \, \mathrm{d}\tau = -\int_{t+\frac{1}{2}}^{t-\frac{1}{2}} f(\tau') \mathrm{d}\tau' = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} f(\tau') \mathrm{d}\tau'$$

Notice that change the variable will also change the interval.

Some graphics will be added here.

4 Discrete Fourier Transform

Continuous Fourier transform has great value in analytical study of signal, but in real world we can only deal with discrete signals in computers. Notation will be different, we use $x[n]$ for discrete-time signal, and $x(t)$ for continuous-time signal. We will start from periodic signals to nonperiodic signals just as we did in continuous case.

4.1 Periodic Signals

First we have a conintuous signal $x(t)$ which has period $2l$, we use machine to measure the signal, of course we cannot measure every instant (if we can, there is no need to study discrete signals any more). So unfortuantely we cannot use

$$c_k = \frac{1}{2l}\int_0^{2l} x(t)e^{-ik\frac{\pi}{l}t}\,\mathrm{d}t, \tag{18}$$

because we need to know $x(t)$ for every possible t. But we can approximate this integral by a Riemann sum.

Suppose we measure $x(t)$ at equally spaced value of t for N times. We pick t at

$$t = \frac{2l}{N}, 2\frac{2l}{N}, 3\frac{2l}{N}, \dots, N\frac{2l}{N}.$$

For t in the any interval between two measure points,

Reserve a picture for discrete signal Riemann sum

Figure 13: discrete signal Riemann sum

$$n\frac{2l}{N} \le t \le (n+1)\frac{2l}{N}$$

Notice the integral (18) above, $x(t)e^{-ik\frac{\pi}{l}t}$ is actually a function of t, we can feed it all measure points to calculate function values. Say we pick a measure point $t = n\frac{2l}{N}$,

$$x(n\frac{2l}{N})e^{-ik\frac{\pi}{l}n\frac{2l}{N}} = x(n\frac{2l}{N})e^{-ik\pi n\frac{2}{N}} = x(n\frac{2l}{N})e^{-2\pi i\frac{kn}{N}}$$

Then we approximate the integral over $n\frac{2l}{N} \le t \le (n+1)\frac{2l}{N}$, this is just the segment area of whole integral. We approximate this area by using area of a rectangle under it, which is

$$x(n\frac{2l}{N})e^{-2\pi i\frac{kn}{N}}\frac{2l}{N}$$

$\frac{2l}{N}$ is the width of the rectangle. Finally we can use a Riemann sum to approximate the integral, then

$$c_k \approx \frac{1}{2l}\sum_{n=1}^{N} x(n\frac{2l}{N})e^{-2\pi i\frac{kn}{N}}\frac{2l}{N} = \frac{1}{N}\sum_{n=1}^{N} x(n\frac{2l}{N})e^{-2\pi i\frac{kn}{N}}$$

We will change notation slightly in order to be symmetric to continuous Fourier transform, set $c_k \approx \hat{x}[k]$, and $x[n] = x(n\frac{2l}{N})$. Then

$$\hat{x}[k] = \frac{1}{N}\sum_{n=1}^{N} x[n]e^{-2\pi i\frac{kn}{N}} \tag{19}$$

This is what we called *discrete Fourier transform*, quite unimpressive, right?

Of course it has a mirror side just as continuous case does, *inverse discrete Fourier transform*,

$$x[n] = \sum_{k=1}^{N} \hat{x}[k]e^{2\pi i\frac{nk}{N}} \tag{20}$$

Although we won't show how it is derived, but we can show why it holds. Do you still remember the when we studied the continuous Fourier transform, we define $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t$, we do not really show how it is derived, but simply defined it and derive the other one. So here we are following the same logic pattern work through them.

Now we will show when (19) holds (20) must be true. We substitute $\hat{x}[k]$ into (20),

$$x[n] = \sum_{k=1}^{N} e^{2\pi i\frac{nk}{N}}\hat{x}[k] = \sum_{k=1}^{N} e^{2\pi i\frac{nk}{N}}\frac{1}{N}\sum_{n=1}^{N} x[n]e^{-2\pi i\frac{kn}{N}}$$

However, we need to rename the variable, because n in the rightmost function is different from n in the leftmost term.

$$\begin{aligned} x[n] &= \sum_{k=1}^{N} e^{2\pi i \frac{nk}{N}} \frac{1}{N} \sum_{n'=1}^{N} x[n'] e^{-2\pi i \frac{kn'}{N}} \\ &= \frac{1}{N} \sum_{n'=1}^{N} \sum_{k=1}^{N} e^{2\pi i \frac{nk}{N}} e^{-2\pi i \frac{kn'}{N}} x[n'] \\ &= \frac{1}{N} \sum_{n'=1}^{N} \sum_{k=1}^{N} e^{2\pi i \frac{k(n-n')}{N}} x[n'] \\ &= \frac{1}{N} \sum_{n'=1}^{N} x[n'] \sum_{k=1}^{N} e^{2\pi i \frac{k(n-n')}{N}} \\ &= \frac{1}{N} \sum_{n'=1}^{N} x[n'] \boxed{\sum_{k=1}^{N} \left(e^{2\pi i \frac{(n-n')}{N}} \right)^{k}} \end{aligned}$$

We need to study a little bit about the term in the box. If $n' = n$, then

$$\sum_{k=1}^{N} \left(e^{2\pi i \frac{(n-n')}{N}} \right)^{k} = \sum_{k=1}^{N} e^{0} = \sum_{k=1}^{N} 1 = N$$

4.2 Properties of Discret Fourier Transform

These properties are similar to those in continuous Fourier transform.

4.2.1 Time shifting

Let $x[n]$ be a discrete-time signal of period N[10]. n_0 is any integer, we define $y[n] = x[n - n_0]$, and its discrete Fourier transform is

$$\begin{aligned} \hat{y}[k] &= \frac{1}{N} \sum_{n=1}^{N} y[n] e^{-2\pi i \frac{kn}{N}} \\ &= \frac{1}{N} \sum_{n=1}^{N} x[n - n_0] e^{-2\pi i \frac{kn}{N}} \end{aligned}$$

[10] It means the siginal will come back to the same value every N measured point (sampling point) after.

Substitute $m = n - n_0$,

$$
\begin{aligned}
&= \frac{1}{N} \sum_{m+n_0=1}^{N-n_0} x[m] e^{-2\pi i \frac{k(m+n_0)}{N}} \\
&= \frac{1}{N} \sum_{m=-n_0+1}^{N-n_0} x[m] e^{-2\pi i \left(\frac{km}{N} + \frac{kn_0}{N}\right)} \\
&= \frac{1}{N} \sum_{m=-n_0+1}^{N-n_0} x[m] e^{-2\pi i \frac{km}{N}} e^{-2\pi i \frac{kn_0}{N}} \\
&= e^{-2\pi i \frac{kn_0}{N}} \left(\frac{1}{N} \sum_{m=-n_0+1}^{N-n_0} x[m] e^{-2\pi i \frac{km}{N}} \right)
\end{aligned}
$$

Here is the idea, notice

$$\sum_{m=-n_0+1}^{N-n_0},$$

$N - n_0 + n_0 - 1 = N - 1$, then we can change the interval and keep the summation the same only if we keep the sampling points to be $N - 1$. So

$$\sum_{m=1}^{N}$$

will also do. Thus,

$$
\begin{aligned}
\hat{y}[k] &= e^{-2\pi i \frac{kn_0}{N}} \left(\frac{1}{N} \sum_{m=1}^{N} x[m] e^{-2\pi i \frac{km}{N}} \right) \\
&= e^{-2\pi i \frac{kn_0}{N}} \hat{x}[k]
\end{aligned}
$$

4.2.2 Parseval's Relation

Follow the steps we did for continuous case, this is quick,

$$\begin{aligned}
\sum_{k=1}^{N} |\hat{x}[k]|^2 &= \sum_{k=1}^{N} \overline{\hat{x}[k]}\hat{x}[k] \\
&= \sum_{k=1}^{N} \left(\frac{1}{N} \sum_{n=1}^{N} \overline{x[n]e^{-2\pi i \frac{kn}{N}}} \right) \hat{x}[k] \\
&= \sum_{k=1}^{N} \left(\frac{1}{N} \sum_{n=1}^{N} \overline{x[n]}e^{2\pi i \frac{kn}{N}} \right) \hat{x}[k] \\
&= \frac{1}{N} \sum_{n=1}^{N} \overline{x[n]} \left(\sum_{k=1}^{N} e^{2\pi i \frac{kn}{N}} \hat{x}[k] \right) \\
&= \frac{1}{N} \sum_{n=1}^{N} \overline{x[n]}x[n] \\
&= \frac{1}{N} \sum_{n=1}^{N} |x[n]|^2
\end{aligned}$$

4.3 Aperiodic Signals

Same procedure will be implemented as the continuous case section 2.2. Again for simplicty, we develop a discrete function that $x[n] = 0$ for $|n| \geq \frac{N}{2}$, where N is period, and an even number. We can get a disrete Fourier transform for the part of $x[n]$ with $|n| \leq \frac{N}{2}$ by using periodic extension. So we define a function

$$\begin{aligned}
&x_N[n] = x[n] \text{ for } 1\frac{N}{2} < n \leq \frac{N}{2} \\
&x_N[n] \text{ has period N}
\end{aligned}$$

Then discrete Fourier transform pair are

$$x_N[n] = \sum_{-\frac{N}{2} < k \leq \frac{N}{2}} \widehat{x_N}[k] e^{2\pi i \frac{nk}{N}} \qquad \widehat{x_N}[k] = \frac{1}{N} \sum_{-\frac{N}{2} < k \leq \frac{N}{2}} x[n] e^{-2\pi i \frac{nk}{N}} \tag{21}$$

We define k^{th} frequency to be $\omega_k = 2\pi \frac{k}{N}$, then

$$\omega_k - \omega_{k-1} = 2\pi \frac{k}{N} - 2\pi \frac{k-1}{N} = \frac{2\pi}{N} = \Delta\omega$$

And also define

$$\hat{x}(\omega)[k] = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n}.$$

Then,

$$\widehat{x_N}[k] = \frac{1}{N} \sum_{-\frac{N}{2}<n\leq\frac{N}{2}} x[n]e^{-2i\pi\frac{nk}{N}}$$

$x[n] = 0$ for all $|n| \geq \frac{N}{2}$, it won't make a difference whether we sum up over interval $[-\frac{N}{2}, \frac{N}{2}]$ or $(-\infty, \infty)$, since all zero terms won't count,

$$\begin{aligned}
\widehat{x_N}[k] &= \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n]e^{-i2\pi\frac{k}{N}n} \\
&= \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega_k n} \\
&= \frac{1}{N}\hat{x}(\omega_k)
\end{aligned}$$

because $N = \frac{2\pi}{\Delta\omega}$,

$$= \frac{1}{2\pi}\hat{x}(\omega_k)\Delta\omega$$

Substitute the result above into $x[n]$,

$$x[n] = x_N[n] = \sum_{-\frac{N}{2}<k\leq\frac{N}{2}} \frac{1}{2\pi}\hat{x}(\omega_k)\Delta\omega e^{2\pi i\frac{nk}{N}} = \frac{1}{2\pi} \sum_{-\frac{N}{2}<k\leq\frac{N}{2}} \hat{x}(\omega_k)e^{2\pi i\frac{nk}{N}}\Delta\omega$$

We will modify the summation restriction like this,

$$\frac{2\pi}{N}(-\frac{N}{2} < k \leq \frac{N}{2}) = -\pi < \omega_k \leq \pi,$$

obviously you can see that we are intentionally ω in summation restriction. Thus, we have

$$\frac{1}{2\pi} \sum_{-\pi<\omega_k\leq\pi} \hat{x}(\omega_k)e^{i\omega_k n}\Delta\omega$$

But note that the argument of the summation is still k, not ω. Now we could rightfully approximate the Riemann sum to integral by letting $N \to \infty$.

Then $\Delta\omega = \frac{2\pi}{N} = 0$. We conclude that

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(\omega)\,\mathrm{d}\omega \qquad \hat{x}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n} \tag{22}$$

$\hat{x}(\omega)$ is the discrete Fourier transform of $x[n]$, and is given a name 'spectrum' of $x[n]$. Compare (22) and (3.2).

5 z-transform

5.1 Linear, Time Invariant Systems

Linear, time invariant (LTI) system has two version, continuous and discrete. Most of differential equation you learned are continuous LTI. The system is linear, because if we feed system with a signal $ax_1 + bx_2$, we receive a signal $ay_1 + by_2$. Time invariant means that for any time shifted signal $x(t-s)$, generates the time shifted output $y(t-s)$. As you no doubt have guessed, difference equations are discrete LTI system. We will present examples later.

5.2 Impulse Response Function

We have seen Dirac delta function,

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

It simply models the idea that there is a unit impulse when time hits 0. We denote $h[n]$ the response from the unit impulse at time 0, naturally it is our *impulse response function.* We need to modify Dirac delta funtion to make it more general,

$$\delta_k[n] = \begin{cases} 1 & \text{if } n = k \text{ and } k > 0 \\ 0 & \text{if } n \neq k \end{cases}$$

it becomes a unit impulse at time k, rest of instants are 0. According to time shift property of all functions, $\delta_k[n] = \delta[n-k]$, accordingly impulse response function will become $h[n-k]$ due to LTI time invariant property, because time shifted input $\delta[n-k]$ generates time shifted $h[n-k]$ output.

Say we have an input signal $x[3]$, the signal is input at time instant 3, we have a crucial observation,

$$x[3] = \cdots + a_1\delta_1[3] + a_2\delta_2[3] + a_3\delta_3[3] + a_4\delta_4[3] + a_5\delta_5[3]\cdots$$

Besides $a_3\delta_3[3] = a_3$, in general $x[k] = a_k$, rest of terms are zero. Why do we bother to write this? Because this is a linear combination,

$$x[n] = \sum_{k=-\infty}^{k=\infty} a_k\delta_k[n]$$

We have seen that the output corresponding to $\delta_k[n]$ is $h[n-k]$. So the output corresponding $x[n]$, by linearity of LTI,

$$y[n] = \sum_{k=-\infty}^{k=\infty} a_k h[n-k] = \sum_{k=-\infty}^{k=\infty} x[k]h[n-k] = (h*x)[n] \tag{23}$$

This is *discrete form of convolution*, compare with (15).

5.3 z-transform

One important feature of discrete-time LTI systems is that all *exponential signals* are *eigenfunctions* for all LTI systems. It will be clear soon. If z is arbitrary complex number, then we define a signal $x[n] = z^n$, which is an complex exponential signal. Feed the signal to LTI system with impulse function $h[n]$,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = H(z)z^n$$

This is called z-**transform** of the impulse response $h[n]$. z^n is eigenfunction and $H(z)$ is eigenvalue, where

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}.$$

Where z^n is feed into a LTI system, result is just z^n multiply a constant, $H(z)$, which is independent of time n.