Fourier Analysis

Yiqian Lu

November 22, 2013

First, we will see some examples.

1. $||x|| = \sum_{j=1}^{\infty} |x_j|^2$
   l^2: let $e_n = \{\underbrace{0, \cdots, 0}_{n-1}, 1, 0, \cdots\}$
   $(e_n, e_m) = S_n m$
   $x^{(n)} = (x_1, \cdots, x_n, 0, \cdots)$
   Then (e_n) is an o.n. basis.

2. $H = L^2(-\pi, \pi) = L^2(\mathbb{T})$.
   $\mathbb{T} \approx \frac{\mathbb{R}}{2\pi\mathbb{Z}}$
   $(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)} dt$

Let $e_n(t) = e^{int}, n = 0, \pm 1, \pm 2, \cdots$
Then $\{e_n : n \in \mathbb{Z}\}$ is an o.n. basis of $L^2(\mathbb{T})$
$\widehat{f(n)} = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$, which is the nth Fourier coefficient.
we call $\sum_{-\infty}^{\infty} \widehat{f(n)} e^{inx}$ the Fourier sequence of f.
Let $A_n(f) = \sum_{-n}^{n} \widehat{f(k)} e^{ikx}$
Let $\sigma_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} A_k(f)$
Then we have

$$(A_n(f))(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{i=-n}^{n} e^{ik(x-t)} f(t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x-t) f(t) dt$$

where

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt}$$

We call it Dirichlet Kernel.

$$\sigma_n f(x) = \frac{1}{2\pi n} \int_{-\pi}^{\pi} \left(\sum_{k=0}^{n-1} D_k(x-t) \right) f(t) dt$$

where

$$k_n(t) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(t)$$

$$\sigma_n f(x) = \frac{1}{2\pi n} \int_{-\pi}^{\pi} k_n(t) f(t) dt$$

we call it Kejer's kernel.

Notice

$$D_n(t) = 1 + 2(\cos t + \cos 2t + \cdots + \cos nt) = \frac{\sin((n+\frac{1}{2})t)}{\sin\frac{t}{2}}$$

$$k_n(t) = \frac{1}{n}\left(\sin\frac{t}{2} + \sin\frac{3t}{2} + \cdots + \sin\frac{(2n-1)t}{2}\right)\frac{\sin\frac{t}{2}}{\sin^2\frac{t}{2}} = \frac{1}{n}\frac{\sin^2(n+\frac{1}{2})t}{\sin^2\frac{t}{2}}$$

And we notice if $\delta > 0$, then $\lim_{n\to\infty}\sup_{|t|\geq\delta} k_n(t) = 0$

$(k_n(t) \leq \frac{1}{n}\frac{1}{(\sin^2\frac{\delta}{2})} \to 0$ as $n \to \infty)$

Theorem. Let $f \in C(-\pi, \pi)$ and $f(-\pi) = f(\pi)$, then $\sigma_n f \to f$ uniformly

Corollary. (Weierstrass) If f is a continuous on $[a, b]$ and $\epsilon > 0$, then $\exists$ a polynomial $p(x)$ s.t. $|f(x) - p(x)| < \epsilon, \forall x \in [a, b]$

Theorem. $\{e_n\}_{n=-\infty}^{\infty}$ is an o.n. basis of $L^2(-\pi, \pi)$ where $e_n(x) = e^{inx}$

Proof. Let $g \in L^2(\mathbb{T})$, then $\exists f \in C(\mathbb{T})$ s.t. $||g - f||_2 < \epsilon$.

By the above theorem $\exists \sigma_n f$ s.t. $||\sigma_n f - f|| < \epsilon$

then

$$||g - \sigma_n f|| \leq ||g - f||_2 + ||f - \sigma_n f||_2 \leq |g - f||_2 + ||f - \sigma_n f||_\infty \leq 2\epsilon$$

$\widehat{f(n)} = (f, e_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$

We have

$$f \sim \sum_{n\in\mathbb{Z}} \widehat{f(k)}e^{ikx}$$

and

$$S_n = \sum_{k=-n}^{n} \widehat{f(k)}e^{ikx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x-t)f(t)dt$$

$$D_n = \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}$$

$$\sigma_n f = \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(x-t)f(t)dt$$

$$k_n(t) = \frac{1}{n}\left(\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\right)^2$$

□

Theorem. If $f \in C(\mathbb{T})$, then $||\sigma_n f - f||_\infty \to 0$ as $n \to \infty$.

Proof. Notice

1. $k_n(t) \geq 0$
2. $\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(t)dt = 1$
3. for given $\delta > 0, \lim_{n\to\infty}\sup_{\delta<|t|\leq\pi} k_n(t) = 0$

Then we have

$$\begin{aligned}
&|\sigma_n f(x) - f(x)| \\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(t)f(x-t)dt - f(x)\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(t)dt\right| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(t)|f(x-t) - f(x)|dt \\
&\leq \frac{1}{2\pi}\int_{|t|\leq\delta} k_n(t)|f(x-t) - f(x)|dt + \frac{1}{2\pi}\int_{\delta<|t|\leq\pi} k_n(t)|f(x-t) - f(x)|dt
\end{aligned}$$

let $\epsilon > 0$ be given, choose $\delta > 0$ s.t. $|f(x-t) - f(x)| < \epsilon, \forall x$.

So

$$\frac{1}{2\pi}\int_{|t|\leq\delta} k_n(t)|f(x-t)-f(x)|dt < \epsilon\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(t)dt = \epsilon$$

Choose N, $\forall n > N$, then $\sup_{\pi\geq|t|\geq\delta} k_n(t) < \epsilon$

$$\frac{1}{2\pi}\int_{\delta<|t|\leq\pi} k_n(t)|f(x-t)-f(x)|dt < 2||f||\epsilon$$

Finally we have

$$|\sigma_n f(x) - f(x)| < (2||f||+1)\epsilon$$

□

Theorem. $\{e_n\}_{n\in\mathbb{Z}}$ is an o.n. basis on $L^2(\mathbb{T})$, so $||f||_2 = ||\hat{f}||_2$

Theorem. If $f \in L^2(\mathbb{T})$, then $||S_n f - f||_2 \to 0$

Theorem.(Classical Riesz-Fisher) Assume $\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$, then $\exists f \in L^2(\mathbb{T})$ s.t. $\widehat{f(n)} = c_n, n \in \mathbb{Z}$.

Theorem.(Parseval Equality) If $f, g \in L^2(\mathbb{T})$, then

$$\sum_{n\in\mathbb{Z}} \widehat{f(n)}\overline{\widehat{g(n)}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$$

We have several other conclusions.

Let $f \in L^1(-\pi,\pi)$, then $\widehat{f(x)}$ is well-defined. It can be compaired

1. $||\sigma_n f - f||_1 \to 0, \forall f \in L^1(\pi)$
2. $\exists f \in L^1(\mathbb{T})$ s.t. $||S_n f - f|| \nrightarrow 0$
3. (Kolmogorov 1922) $\exists f \in L^1(\mathbb{T})$ s.t.$\{\delta_n f(x)\}$ is diverged for a.e. x
4. (L. Carleson 1966) If $f \in Ł^2(\mathbb{T})$, then $S_n f(x)$ converges a.e.