Fourier Transforms and Linear Filters

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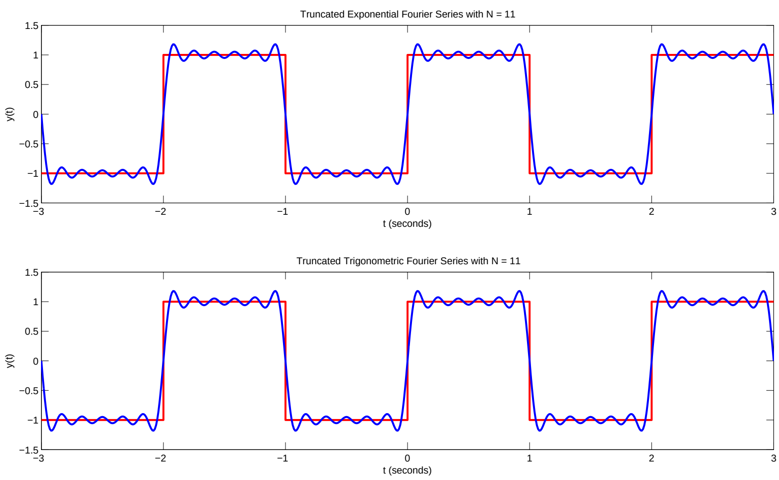

Abstract

This note presents you the essential theoretical knowledge about Fourier series and transform, which paves the way to linear filter. Considering most of economics student seldom have been trained on complex analysis and Fourier transform, we will devote relatively large proportion to those knowledge. Then we come to discuss the linear filter, such as Hodrick-Prescott filter, band-pass filter and etc. This note should be taken as the general training for macroeconomics student, not only for DSGE empirical study.

Chapter 1

Complex Numbers

1.1 Definition and Basic Operations

A **complex number** z is given in the form

$$z = a + bi.$$

a is called the **real part**, and b the **imaginary part**, corresponding notations are $\operatorname{Re} z$ and $\operatorname{Im} z$. i is the imaginary number such that $i^2 = -1$. We denote complex number set by $\mathbb{C}$.

Operations of addition and multiplication is similar to scalar and vector operation,

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = ac + (ad)i + (bc)i - bd = (ac - bd) + (ad + bc)i$$

For example,

$$(5 + 2i) + (2 + 3i) = 7 + 5i$$
$$(5 + 2i)(2 + 3i) = 10 + 15i + 4i - 6 = 4 + 19i$$

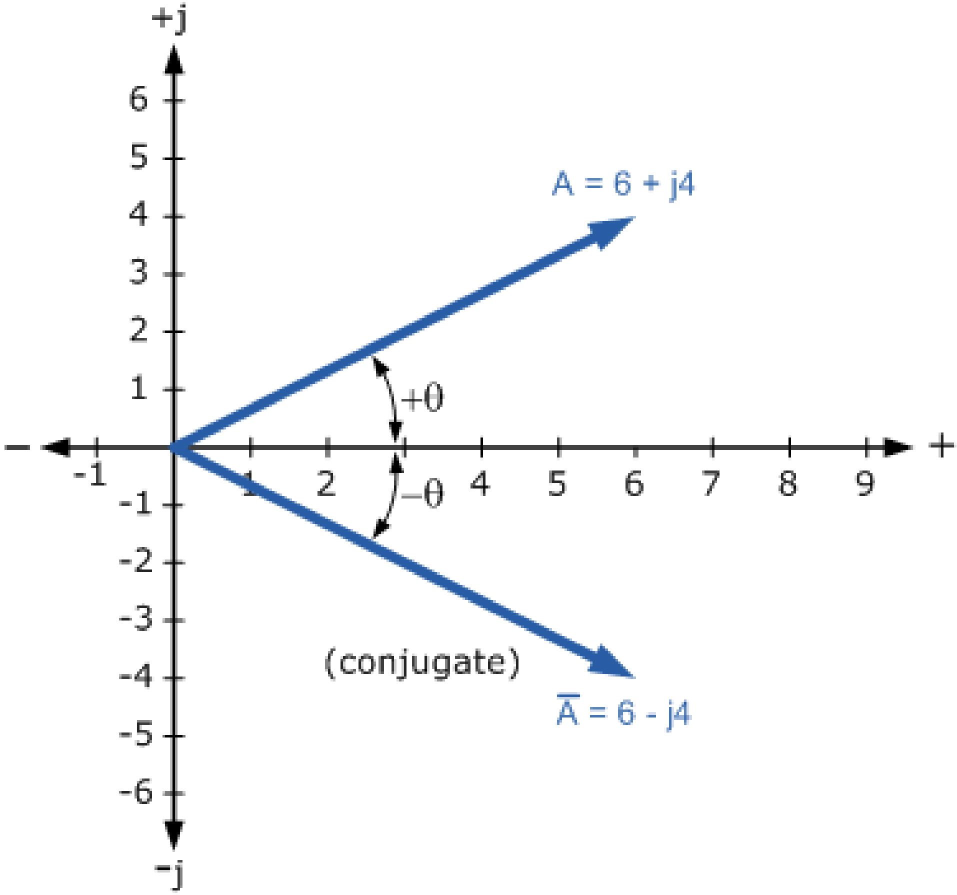

Figure 1.1: Complex conjugate

Conjugate of $z = a + bi$ defined by

$$\bar{z} = a - bi.$$

We also have a coordinate for complex number system,

the horizontal axis is **real axis**, the vertical axis is **imaginary axis**. Conjugate of z is the mirror image of real axis. Actually complex addition operation is a resemblance of vector addition, if you view $z = a + bi$ as a vector $\begin{bmatrix} a \\ b \end{bmatrix}$. A number is real, if and only if its conjugate is itself.

We also have another resemblance here, the **modulus**, which is the same idea of **norm** used in linear algebra. And $\theta = \tan^{-1}\frac{b}{a}$ is called the **argument** of z. Recall the norm of a vector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}^\top \boldsymbol{v}} = \sqrt{v_1^2 + v_2^2}.$$

Modulus is defined by,

$$\|z\| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

We can verify the second equation,

$$z\bar{z} = (a + bi)(a - bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2$$

Another property of modulus,

$$\|z_1 z_2\| = \|z_1\| \|z_2\|,$$

which is easy to verify, since $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

$$\begin{aligned} \|z_1 z_2\| &= \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2} \\ &= \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} \\ &= \|z_1\| \|z_2\| \end{aligned}$$

Besides, addition and multiplication follow algebraic rules,

$$\begin{aligned} z_1 + z_2 &= z_2 + z_1 & z_1 z_2 &= z_2 z_1 \\ z_1(z_2 + z_3) &= z_1 z_2 + z_1 z_3 & (z_1 + z_2) z_3 &= z_1 z_3 + z_2 z_3 \\ z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 & z_1(z_2 z_3) &= (z_1 z_2) z_3 \end{aligned}$$

1.2 Polar Coordinates and Multiplication

We have mentioned that the complex addition resembles vector addition, however multiplication has not such straightforward correspondence. However, if we use polar coordinates, complex multiplication is very intuitively understandable.

Suppose θ is the angle between real axis and point (a, b), we call it **argument** of z. From trigonometric knowledge,

$$a = \|z\| \cos\theta \qquad b = \|z\| \sin\theta$$

and so

$$z = \|z\| \cos\theta + \|z\| \sin\theta i = \|z\| (\cos\theta + i \sin\theta)$$

Of course we need another complex number to perform multiplication, say w,

$$w = \|w\| (\cos\phi + i \sin\phi).$$

Thus

$$\begin{aligned} zw &= \|z\| (\cos\theta + i \sin\theta) \|w\| (\cos\phi + i \sin\phi) \\ &= \|z\| \|w\| [\cos\theta \cos\phi + i \cos\theta \sin\phi + i \cos\phi \sin\theta + i^2 \sin\theta \sin\phi] \\ &= \|z\| \|w\| [\cos\theta \cos\phi + i \cos\theta \sin\phi + i \cos\phi \sin\theta - \sin\theta \sin\phi] \\ &= \|z\| \|w\| [\cos\theta \cos\phi - \sin\theta \sin\phi + i(\cos\theta \sin\phi + \cos\phi \sin\theta)]. \end{aligned}$$

High school math shall be made use of, recall the *addition formula of trigonometry*,

$$\sin(x + y) = \sin x \cos y + \sin y \cos x$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

You can verify that

$$zw = \|z\| \|w\| [\cos(\theta + \phi) + i \sin(\theta + \phi)]$$

To emphasise more, there is a logical distinction between multiplication in $\mathbb{R}^2$ and $\mathbb{C}$. A vector can not multiple another vector in $\mathbb{R}^2$ and results a vector [1], but in $\mathbb{C}$ a complex number is a always a result of complex multiplication.

[1] Dot product does not count whic results a scalar.

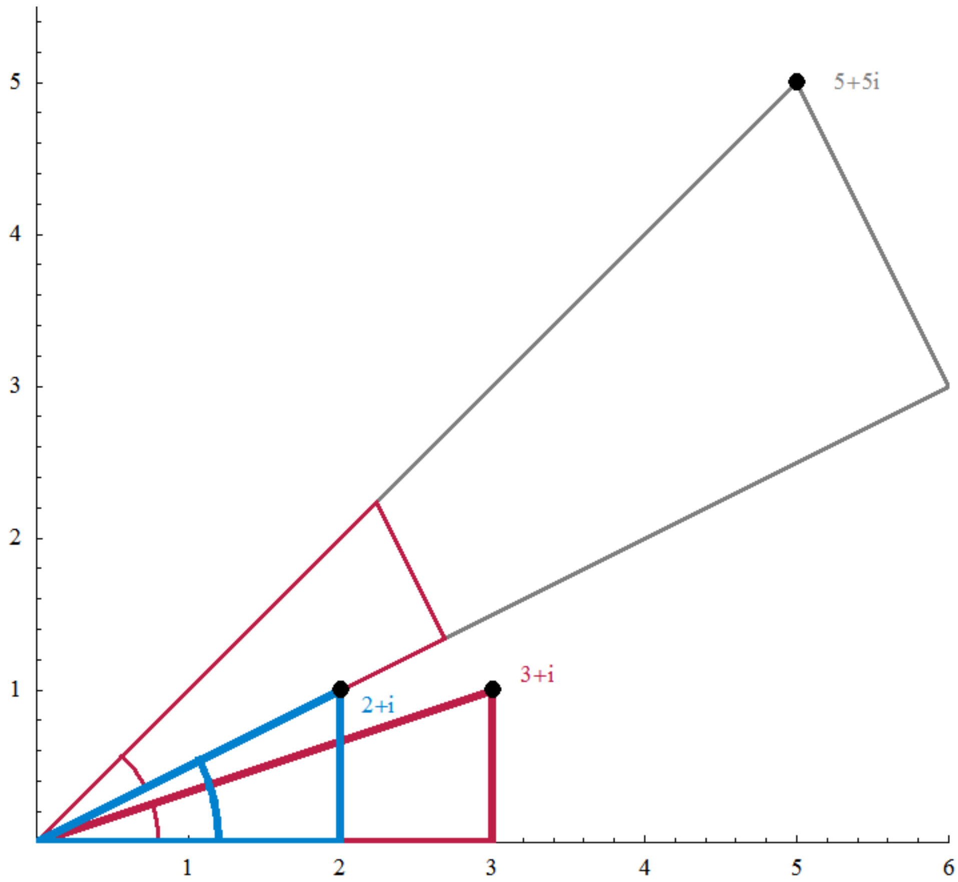

Figure 1.2: Complex multiplication

Look at figure (1.2), make sure you understand the machenism.

1.3 De Moivre's Theorem

De Moivre's Theorem is easily understandable, it is used for calculating powers of complex number in polar coordinates. It says

$$z^k = r^k(\cos k\theta + i \sin k\theta),$$

r is a positive number, such as $\|z\|$ or $\|w\|$. We can verify this equation, say we have $z = w = r(\cos\theta + i\sin\theta)$.

$$\begin{aligned} zw = zz &= r^2(\cos\theta + i\sin\theta)^2 \\ &= r^2(\cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta) \\ &= r^2(\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta) \end{aligned}$$

By using the famous trigonometric identity $\sin^2 x + \cos^2 x = 1$,

$$\begin{aligned} zz &= r^2(\cos^2\theta + 2i\cos\theta\sin\theta - (1 - \cos^2\theta)) \\ &= r^2(2\cos^2\theta - 1 + 2i\cos\theta\sin\theta) \end{aligned}$$

By using *double angle formula*, $\cos 2x = 2\cos^2 x - 1$, yields

$$zz = r^2(\cos 2\theta + 2i\cos\theta\sin\theta)$$

Last step, by using *trigonometric product formula*, $\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$, we have

$$\begin{aligned} zz &= r^2(\cos 2\theta + 2i\frac{1}{2}[\sin(\theta+\theta) + \sin(\theta-\theta)]) \\ &= r^2(\cos 2\theta + i\sin 2\theta) \end{aligned}$$

Same as zzz,

$$\begin{aligned} zzz &= r^2(\cos 2\theta + i\sin 2\theta)r(\cos\theta + i\sin\theta) \\ &= r^3(\cos 3\theta + i\sin 3\theta) \end{aligned}$$

As you can see, trigonometric functions and identities are useful deriving theorems, you don't need to memorize them, but you should be familiar with them, then you will know when shall you use an identity to help you reach the goal.

1.4 Complex Exponential

In this section we will discuss complex exponential, the most interesting part in complex numbers. It connects seemingly unrelated knowledge, helps you see the beauty of mathematics for real. And the pace is a little fast, but still easily understandable.

First recall *Taylor expansion* (however, we are actually using Maclaurin series here),

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \dots,$$

which e^a is expanded as an infinite series. Also recall Taylor expansion of $\cos a$ and $\sin a$,

$$\cos a = 1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \dots$$
$$\sin a = a - \frac{a^3}{3!} + \frac{a^5}{5!} - \frac{a^7}{7!} + \dots$$

Your hunch might tell you $\cos a$ and $\sin a$ have close relation with e^a.

Now let's turn these equipment to our own use for complex exponential, we have a famous equation in complex exponential,

$$e^x e^{iy} = e^x(\cos y + i \sin y) = e^x \cos y + e^x i \sin y.$$

Clearly, our work here is to show $e^{iy} = \cos y + i \sin y$, thus we need to apply Taylor expansion to e^{iy} to study the inner mechanism,

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \dots \\ &= 1 + iy + \frac{i^2y^2}{2!} + \frac{i^3y^3}{3!} + \dots \end{aligned}$$

This isn't over, let's select *odd terms* and put then into a subseries,

$$\begin{aligned} &1 + \frac{i^2y^2}{2!} + \frac{i^4y^4}{4!} + \dots \\ &= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \dots \end{aligned}$$

Isn't this the Taylor expansion of $\cos y$? Similarly, the *even terms*, as you might have guessed,

$$\begin{aligned} & iy + \frac{i^3y^3}{3!} + \frac{i^5y^5}{5!} + \dots \\ & = i(y - \frac{y^3}{3!} + \frac{y^5}{5!} + \dots) \\ & = i \sin y \end{aligned}$$

Thus we have shown that

$$e^{iy} = \cos y + i \sin y. \tag{1.1}$$

So

$$e^x e^{iy} = e^x(\cos y + i \sin y) = e^{x+iy} = e^x \cos y + ie^x \sin y \tag{1.2}$$

We implicitly admit the algebraic rule that also hold on complex exponentials,

$$e^{z_1} e^{z_2} = e^{z_1 z_2}$$

This is easy to derive, just set $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, use formula (1.2), we have $e^{z_1} = e^{a_1+b_1 i} = e^{a_1} \cos b_1 + ie^{a_1} \sin b_1$ and $e^{z_2} = e^{a_2+b_2 i} = e^{a_2} \cos b_2 + ie^{a_2} \sin b_2$. And multiply them, result comes.

1.5 Complex Exponential and Trigonometry

There is a frequently used form of writing a complex number. The modulus of $x + iy$ is r, which gives $x = r \cos \theta$ and $y = r \sin \theta$, and

$$x + iy = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

For the rest of this section, we are going to take a close look at relationship of trigonometry and complex exponential. A slightly notation change won't damage the essence, say we have an angle θ, so

$$e^{i\theta} = \cos \theta + i \sin \theta, \tag{1.3}$$

this is the one of the most famous formula ever, **Euler's formula**. It is a beautiful since it put several seemingly unrelated things together. And its conjugate,

$$e^{-i\theta} = \cos\theta - i\sin\theta \tag{1.4}$$

Take them as a linear equation system, use matrix form,

$$\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} e^{i\theta} \\ e^{-i\theta} \end{bmatrix}$$

Solve for $\cos\theta$ by $\sin\theta$,

$$\begin{aligned} \cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \end{aligned}$$

Set another angle ϕ, also $\cos\phi$ and $\sin\phi$. We multiply $\cos\theta$ and $\sin\phi$ to see what's going next,

$$\begin{aligned} \cos\theta\sin\phi &= \frac{1}{2}(e^{i\theta} + e^{-i\theta})\frac{1}{2i}(e^{i\phi} - e^{-i\phi}) \\ &= \frac{1}{4i}(e^{i\theta} + e^{-i\theta})(e^{i\phi} - e^{-i\phi}) \\ &= \frac{1}{4i}(e^{i\theta}e^{i\phi} - e^{i\theta}e^{-i\phi} + e^{-i\theta}e^{i\phi} - e^{-i\theta}e^{-i\phi}) \\ &= \frac{1}{4i}(e^{i(\theta+\phi)} - e^{i(\theta-\phi)} + e^{i(\phi-\theta)} - e^{-i(\theta+\phi)}) \end{aligned}$$

Use formula (1.3) and (1.4),

$$\begin{aligned} \cos\theta\sin\phi =& \frac{1}{4i}[\cos(\theta+\phi) + i\sin(\theta+\phi) - \cos(\theta-\phi) - i\sin(\theta-\phi) \\ & + \cos(\phi-\theta) + i\sin(\phi-\theta) - \cos(\theta+\phi) + i\sin(\theta+\phi)] \end{aligned}$$

Because $\cos(-a) = \cos a$ and $\sin(-a) = -\sin a$, we can rewrite equation above as,

$$\begin{aligned} \cos\theta\sin\phi =& \frac{1}{4i}[\cos(\theta+\phi) + i\sin(\theta+\phi) - \cos(\phi-\theta) + i\sin(\phi-\theta) \\ & + \cos(\phi-\theta) + i\sin(\phi-\theta) - \cos(\theta+\phi) + i\sin(\theta+\phi)]. \end{aligned}$$

Coloured pair will be canceled, so

$$\begin{aligned}\sin\phi\cos\theta &= \frac{1}{4i}[2i\sin(\phi+\theta)+2i\sin(\phi-\theta)]\\ &= \frac{1}{2}[\sin(\phi+\theta)+\sin(\phi-\theta)]\end{aligned}$$

This is *trigonometric production formula* derived by complex exponential, you can try rest of formulas by yourself, $\sin\phi\sin\theta$ and $\cos\phi\cos\theta$.

Next example we will try $\cos 2\theta$, first we should know that from (1.3), we have

$$e^{i2\theta} = \cos 2\theta + i\sin 2\theta$$

$\cos 2\theta$ is the real part of complex number, which can be denoted

$$\operatorname{Re} e^{i2\theta}.$$

We can rewrite $e^{i2\theta}$ as $(e^{i\theta})^2$, then (1.3) can directly made use of,

$$\operatorname{Re} e^{i2\theta} = \operatorname{Re}(e^{i\theta})^2 = \operatorname{Re}(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta$$

We have just derived *double angle formula*,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Chapter 2

Fourier Analysis

Fourier series is by no means as intuitive as, say Taylor series, in the very beginning. Notations might seem a great mess to you, several superscripts and subscripts bind together. And applied examples seem all come from physics, electronic engineering, computer science, it is tremendously hard for average economics students to make any sense out of it. But here we will present all relevant knowledge in a gentle manner.

2.1 Brief History Background

About two centuries ago, the famous French physicist Joseph Fourier (1768-1830) found that 'every' function[1] could be represented by an infinite trigonometric series resulted from investigation of partial differential equation of heat conduction. Fourier formulated the model describing evolution of the temperature, $T(x,t)$ of a thin wire of length π, in which $\{0 \leq x \leq \pi\}$ and t is time. He also assumed zero temperature at the ends: $T(0,t) = 0$ and

[1] Actually not every function can be.

$T(\pi, t) = 0$. Then

$$f(x) = T(x, 0) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

And he solved b_n by integration,

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, \mathrm{d}x$$

He argued that $T(x, t)$ is a solution of heat equation,

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

with boundary condtion,

$$T(0, t) = T(\pi, t) = 0, \quad t \geq 0.$$

Solution can be represented by

$$T(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$$

which naturally becomes the first application of Fourier series.

2.2 Odd or Even

Some properties of function we should review here, thus no proof will be provided, you take them as facts. They are very straightforward, needless to waste time on proving.

Theorem 2.2.1. *Let $f(x)$ be a funtion of which domain is $[-a, a]$, then*

$$\int_{-a}^{a} f(x) \, \mathrm{d}x = 2 \int_{-a}^{0} f(x) \, \mathrm{d}x = 2 \int_{0}^{a} f(x) \, \mathrm{d}x \tag{2.1}$$

if $f(x)$ is an even function. And

$$\int_{-a}^{a} f(x) \, \mathrm{d}x = 0$$

if $f(x)$ is an odd funtion.

Theorem 2.2.1. *Multiplication and addition rules,*

$$\begin{aligned} odd + odd &= odd \\ odd \times odd &= even \\ even + even &= even \\ even \times even &= even \\ odd \times even &= odd \end{aligned} \tag{2.2}$$

Note that and odd function plus an odd one is still an odd function, which is different from real number addition. Rest of them resemble the real number operation.

2.2.1 Where do we start?

This is a learning notes, it teaches you something you do not know yet, so it would be quite misgiving to illustrate Fourier series with example in the beginning.

We start with a simple question: what functions have a representation which can be express by a summation of $\sin kt$ and $\cos kt$,

$$f(t) = \sum_{k=-\infty}^{\infty} c_k[\cos(kt) + i\sin(kt)] \tag{2.3}$$

i is imaginary number, c_k is a constant. It might looks very strange makes you wondering why on earth would we invent such a linear combination even with imaginary number, the reason is for sake of easy-manipulation. Based on last chapter, it would immediately remind you **Eular formula**,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Then here we have

$$e^{ikt} = \cos(kt) + i\sin(kt).$$

So the question becomes to ask what fucntions $f(t)$ have a representation,

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \tag{2.4}$$

In modern Fourier theory it is common to write Fourier series involving complex exponentials. Notice that for every integer k, $e^{ikt} = \cos(kt) + i\sin(kt)$ is a periodic with period 2π. So if $f(t)$ needs to be expressed by a trigonometric series as right-hand side, $f(t)$ is necessarily be featured with period 2π.

The second question is, if $f(t)$ has period of 2π and be expressed as trigonometric series, then what are coefficients c_k? Actually this seemingly intractable problem is quite easy to handle

To start, recall from your linear algebraic course, that if an inner space product defined on $[-\pi, \pi]$ is

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t = 0.$$

We say $f(t)$ and $g(t)$ are orthogonal. Named *orthogonality condition*, it has a naturally extension on trigonometry or complex exponential

$$\int_{-\pi}^{\pi} e^{ikt}e^{-ijt}\,\mathrm{d}t = 0 \qquad \text{if } j \neq k$$

We can solve this integral,

$$\begin{aligned}\int_{-\pi}^{\pi} e^{ikt}e^{-ijt}\,\mathrm{d}t &= \int_{-\pi}^{\pi} e^{i(k-j)t}\,\mathrm{d}t \\ &= \frac{1}{i(k-j)}e^{i(k-j)t}\Big|_{-\pi}^{\pi} \\ &= \frac{1}{i(k-j)}\left[e^{i(k-j)\pi} - e^{-i(k-j)\pi}\right] \\ &= 0\end{aligned}$$

The last equation uses the fact that

$$\frac{e^{i(k-j)\pi}}{e^{-i(k-j)\pi}} = e^{i(k-j)2\pi} = \cos[(k-j)2\pi] + i\sin[(k-j)2\pi]$$

According to our basic knowledge of trigonometries, we know

$$\cos(\alpha 2\pi) = 1$$
$$\sin(\beta 2\pi) = 0.$$

Then

$$\frac{e^{i(k-j)\pi}}{e^{-i(k-j)\pi}} = 1$$
$$e^{i(k-j)\pi} = e^{-i(k-j)\pi}.$$

In order to use orthogonal condition to eliminate all c_k with $k \neq j$, we multiply e^{-ijt} onto both side of (2.4) and integrate on interval $[\pi, -\pi]$, yields

$$\int_{-\pi}^{\pi} f(t)e^{-ijt}\,\mathrm{d}t = \sum_{k=1}^{\infty} c_k \int_{-\pi}^{\pi} e^{ikt}e^{-ijt}\,\mathrm{d}t$$

Because of orthogonality condition, all terms on the right-hand side with $k \neq j$ are eliminated,

$$\int_{-\pi}^{\pi} f(t)e^{-ijt}\,\mathrm{d}t = c_j \int_{-\pi}^{\pi} e^{ijt}e^{-ijt}\,\mathrm{d}t = c_j \int_{-\pi}^{\pi} \mathrm{d}t = 2\pi c_j$$

Thus, we solve the problem

$$c_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-ijt}\,\mathrm{d}t \tag{2.5}$$

2.3 Fourier Series Expansion

For the rest of this chapters, we are going to see lots of examples rather than pure theory. We have seen that how we can calculate c_k to expand a function into Fourier series, we will walk you through lots of example in order make you feel familiar with it.

Before we start, I shall give the most standard form of Fourier series which you can see from textbook,

$$f(t) = a_0 + \sum_{k=-\infty}^{\infty} (a_k \cos kt + b_k \sin kt) \tag{2.6}$$

This is a bit different from (2.3), the complex exponential form, keep this in mind, it will be clear where it come from later on.

2.3.1 2π Period Expansion

Example 1. *Find the Fourier expansion of*

$$f(t) = \sin(t + \frac{\pi}{4})$$

Actually we do not need to compute c_k directly, we can use Euler's formula to manipulate $f(x)$, recall

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

Thus,

$$\begin{aligned} f(t) &= \frac{1}{2i}\left[e^{i(t+\pi/4)} - e^{-i(t+\pi/4)}\right] \\ &= \frac{1}{2i}\left[e^{it}e^{i(\pi/4)} - e^{-it}e^{-i(\pi/4)}\right] \\ &= \frac{1}{2i}\frac{\sqrt{2}}{2}(1+i)e^{it} - \frac{1}{2i}\frac{\sqrt{2}}{2}(1-i)e^{-it} \end{aligned} \tag{2.7}$$

The last equation makes use of the facts:

$$\begin{aligned} e^{i(\pi/4)} &= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}(1+i) \\ e^{-i(\pi/4)} &= \cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}) = \frac{\sqrt{2}}{2}(1-i) \end{aligned}$$

And (2.7) *is what we called fourier expansion.*

Till now we only derive coefficients in complex exponential form, next we shall follow the standard procedure to derive coefficients in trigonometric form as Fourier himself did. You are not obliged to memorise the procedure, but you will find a common pattern which is considerably easy to understand.

We integrate (2.6)

$$f(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt),$$

out t, on both sides over interval $[\pi, -\pi]$,

$$\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = a_0 \int_{-\pi}^{\pi} \mathrm{d}t + \sum_{k=-\infty}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos kt\,\mathrm{d}t + b_k \int_{-\pi}^{\pi} \sin kt\,\mathrm{d}t \right)$$

The terms in the brackets equal zero, thus

$$\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = 2\pi a_0$$
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = a_0$$

Next we need to find a_k, but this time we cannot directly integrate (2.6), we integrate its modified version,

$$f(t) \cos jt = a_0 \cos jt + \sum_{k=1}^{\infty} (a_k \cos kt \cos jt + b_k \sin kt \cos jt).$$

You simply multiply (2.6) with $\cos jt$ on both sides to get it, for integer $j \geq 1$, then integrate

$$\int_{-\pi}^{\pi} f(t) \cos jt\,\mathrm{d}t = a_0 \int_{-\pi}^{\pi} \cos jt\,\mathrm{d}t + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos kt \cos jt\,\mathrm{d}t \right.$$
$$\left. + \sum_{k=1}^{\infty} b_k \int_{-\pi}^{\pi} \sin kt \cos jt\,\mathrm{d}t \right)$$

We can calculate the first integral on the right-hand side analytically,

$$\int_{-\pi}^{\pi} \cos jt\,\mathrm{d}t = \frac{1}{j} \sin{(jt)} \Big|_{-\pi}^{\pi} = 0$$

The second one, use orthogonality condition[2]

$$\int_{-\pi}^{\pi} \cos kt \cos jt\,\mathrm{d}t = \begin{cases} 0 & \text{if } k \neq j \\ \pi & \text{if } k = j \end{cases}$$

[2] Proof is in my notes *Trigonometry*.

If $k = j$, we can prove

$$\begin{aligned}\int_{-\pi}^{\pi} \cos^2 kt \, \mathrm{d}t &= \int_{-\pi}^{\pi} \frac{\cos 2kt + 1}{2} \, \mathrm{d}t \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \cos 2kt \, \mathrm{d}t + \frac{1}{2} \int_{-\pi}^{\pi} \mathrm{d}t \\ &= \frac{1}{2} \left[\frac{1}{2k} \sin 2kt \right] \Bigg|_{-\pi}^{\pi} + \pi \\ &= \pi\end{aligned}$$

Last is also due to orthogonality condition,

$$\int_{-\pi}^{\pi} \cos kt \sin jt \, \mathrm{d}t = 0, \quad \forall k \neq j$$

Then we only left,

$$\int_{-\pi}^{\pi} f(x) \cos kt \, \mathrm{d}t = a_k \pi.$$

Thus

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, \mathrm{d}t$$

To get b_k is similar with steps above, but multiply both sides of (2.6) by $\sin jt$, for integer $j \geq 1$.

$$f(t) \sin jt = a_0 \sin jt + \sum_{k=1}^{\infty} (a_k \cos kt \sin jt + b_k \sin kt \sin jt)$$

Integrate t out over $[-\pi, \pi]$,

$$\begin{aligned}\int_{-\pi}^{\pi} f(t) \sin jt \, \mathrm{d}t = a_0 \int_{-\pi}^{\pi} \sin jt \, \mathrm{d}t + \sum_{k=1}^{\infty} \Bigg(a_k \int_{-\pi}^{\pi} \cos kt \sin jt \, \mathrm{d}t \\ + b_k \int_{-\pi}^{\pi} \sin kt \sin jt \, \mathrm{d}t \Bigg)\end{aligned}$$

The first integration term is

$$a_0 \int_{-\pi}^{\pi} \sin jt \, \mathrm{d}t = a_0 \left[-\frac{1}{j} \cos jt \right] \Bigg|_{-\pi}^{\pi} = 0$$

The second, follow orthogonality condition,

$$a_k \int_{-\pi}^{\pi} \cos kt \sin jt \, \mathrm{d}t = 0, \forall \quad k, j.$$

The third,

$$b_k \int_{-\pi}^{\pi} \sin kt \sin jt \, \mathrm{d}t = \begin{cases} 0 & \text{if } k \neq j \\ \\ \pi & \text{if} k = j \end{cases}$$

Because if $k = j$, then we have

$$\begin{aligned} b_k \int_{-\pi}^{\pi} \sin^2 kt \, \mathrm{d}t &= b_k \int_{-\pi}^{\pi} \left(\frac{1 - \cos 2kt}{2} \right) \mathrm{d}t \\ &= b_k \left(\frac{1}{2} \int_{-\pi}^{\pi} \mathrm{d}t - \frac{1}{2} \int_{-\pi}^{\pi} \cos 2kt \, \mathrm{d}t \right) \\ &= b_k \pi - b_k \frac{1}{2} \left[\frac{1}{kt} \sin 2kt \right] \Bigg|_{-\pi}^{\pi} \\ &= b_k \pi \end{aligned}$$

Then use all above result, we only left,

$$\int_{-\pi}^{\pi} f(t) \sin kt = b_k \pi$$

Thus,

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, \mathrm{d}t$$

In together, our Fourier series are,

$$f(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \qquad \text{(Fourier 1)}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, \mathrm{d}t \qquad \text{(Fourier 2)}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, \mathrm{d}t \qquad \text{(Fourier 3)}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, \mathrm{d}t \qquad \text{(Fourier 4)}$$

Next we shall use a full example to walk you through a process of Fourier expansion of period 2π.

Example 1. *Find the Fourier series of*

$$f(t) = \left| \sin \frac{x}{2} \right|$$

Use formula (Fourier 2), we get

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sin \frac{t}{2} \right| \mathrm{d}t$$

We know it is an even function and $\sin t/2 \geq 0$ over interval $[0, \pi]$, then we can remove absolute value operator, and use half interval,

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_0^{\pi} \sin \frac{t}{2} \, \mathrm{d}t \\
&= \frac{1}{\pi} \left[-2 \cos \frac{t}{2} \right] \Big|_0^{\pi} \\
&= \frac{1}{\pi} \left(-2 \cos \frac{\pi}{2} + 2 \cos \frac{0}{2} \right) \\
&= \frac{2}{\pi}
\end{aligned}$$

Use formula (Fourier 3), we get

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \left| \sin \frac{t}{2} \right| \cos kt \, \mathrm{d}t$$

Even function multiply even function is an even function,

$$a_k = \frac{2}{\pi} \int_0^{\pi} \sin \frac{t}{2} \cos kt \, \mathrm{d}t$$

We can't use orthogonality condition here, because k is an integer and $k \geq 1$. So use trigonometric product formula, yields

$$\begin{aligned}
a_k &= \frac{1}{\pi} \int_0^{\pi} \left[\sin\left(\frac{t}{2} + kt \right) + \sin\left(\frac{t}{2} - kt \right) \right] \mathrm{d}t \\
&= \frac{1}{\pi} \int_0^{\pi} \left[\sin\left(\frac{1+2k}{2} t \right) + \sin\left(\frac{1-2k}{2} t \right) \right] \mathrm{d}t \\
&= \frac{1}{\pi} \left[-\frac{2}{1+2k} \cos \frac{1+2k}{2} t - \frac{2}{1-2k} \cos \frac{1-2k}{2} t \right] \Big|_0^{\pi} \\
&= \frac{1}{\pi} \left[\frac{4}{1+2k} + \frac{4}{1-2k} \right] \\
&= \frac{1}{\pi} \left(\frac{8}{1-4k^2} \right)
\end{aligned}$$

Similarly, b_k

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \left|\sin\frac{x}{2}\right| \sin kt = 0$$

This is not because of orthogonality, look at integral carefully, recall your high school math again, it is an even function multiply an odd function, which results an odd function. Besides the interval is symmetric on origin, the area on both sides cancel out.

So here is the final result of Fourier expansion,

$$\left|\sin\frac{x}{2}\right| = \frac{2}{\pi} + \sum_{k=1}^{\infty}\frac{1}{\pi}\left(\frac{8}{1-4k^2}\right)\cos kt$$

We can control how close the trigonometric series can approximate function by choosing n,

$$S_n(t) = \left|\sin\frac{x}{2}\right| = \frac{2}{\pi} + \sum_{k=1}^{n}\frac{1}{\pi}\left(\frac{8}{1-4k^2}\right)\cos kt$$

2.3.2 Arbitrary Period Expansion

Above we mainly assume that period is 2π, actually we can modify Fourier series to apply to functions have period $2l$, of which $l \neq \pi$. We can set $\tau = \frac{\pi}{l}t$, but we do not work on τ directly,

$$f(t) = \sum_{k=1}^{\infty} c_k e^{ik\tau} = \sum_{k=1}^{\infty} c_k e^{ik\frac{\pi}{l}t} \tag{2.8}$$

and

$$c_k = \frac{1}{2l}\int_{-l}^{l} f(t)e^{-ik\tau} = \frac{1}{2l}\int_{-l}^{l} f(t)e^{-ik\frac{\pi}{l}t} \tag{2.9}$$

We can check the periodicity,

$$e^{ik\frac{\pi}{l}(t+2l)} = e^{ik\frac{\pi}{l}t}e^{i2k\pi} = e^{ik\frac{\pi}{l}t},$$

by using the fact that

$$e^{i2k\pi} = \cos(2k\pi) + i\sin(2k\pi) = 1.$$

Deriving c_k directly following formula,

$$c_k = \frac{1}{2l}\int_{-l}^{l} f(t)e^{ik\frac{\pi}{l}t}\,\mathrm{d}t$$

$$c_k = \frac{1}{2l}\int_{-l}^{l} e^{-im\frac{\pi}{l}t}e^{ik\frac{\pi}{l}t}\,\mathrm{d}t = \begin{cases} 2l & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}$$

Continue our investigation further, we now try to turn (2.8) into sine and cosine form, use Euler's formula again, the symmetry here will be made use of,

$$e^{ik\frac{\pi}{l}t} = \cos\frac{k\pi t}{l} + i\sin\frac{k\pi t}{l},$$

substitute into (2.8), and we make a slight change on lower limit into negative infinity,

$$f(t) = \sum_{k=-\infty}^{\infty} c_k\left(\cos\frac{k\pi t}{l} + i\sin\frac{k\pi t}{l}\right)$$

First, check the situation when $k = 0$,

$$c_0\left(\cos\frac{0\pi t}{l} + i\sin\frac{0\pi t}{l}\right) = c_0$$

Second, check when $k = 1$ and $k = -1$ together,

$$\begin{aligned} & c_1\left(\cos\frac{1\pi t}{l} + i\sin\frac{1\pi t}{l}\right) + c_{-1}\left(\cos\frac{-1\pi t}{l} + i\sin\frac{-1\pi t}{l}\right) \\ &= c_1\left(\cos\frac{\pi t}{l} + i\sin\frac{\pi t}{l}\right) + c_{-1}\left(\cos\frac{\pi t}{l} - i\sin\frac{\pi t}{l}\right) \\ &= \cos\frac{\pi t}{l}(c_1 + c_{-1}) + \sin\frac{\pi t}{l}(ic_1 - ic_{-1}) \end{aligned}$$

Again, check when $k = 2$ and $k = -2$ together,

$$\begin{aligned} & c_2\left(\cos\frac{2\pi t}{l} + i\sin\frac{2\pi t}{l}\right) + c_{-2}\left(\cos\frac{-2\pi t}{l} + i\sin\frac{-2\pi t}{l}\right) \\ &= \cos\frac{\pi t}{l}(c_2 + c_{-2}) + \sin\frac{\pi t}{l}(ic_2 - ic_{-2}) \end{aligned}$$

As you can guess, $k = 3$ and $k = -3$,

$$\cos\frac{\pi t}{l}(c_3 + c_{-3}) + \sin\frac{\pi t}{l}(ic_3 - ic_{-3})$$

Keep on doing this, and add them up becomes a trigonometric series,

$$
\begin{aligned}
f(t) = c_0 &+ \cos\frac{\pi t}{l}(c_1 + c_{-1}) + \sin\frac{\pi t}{l}(ic_1 - ic_{-1}) \\
&+ \cos\frac{\pi t}{l}(c_2 + c_{-2}) + \sin\frac{\pi t}{l}(ic_2 - ic_{-2}) \\
&+ \cos\frac{\pi t}{l}(c_3 + c_{-3}) + \sin\frac{\pi t}{l}(ic_3 - ic_{-3}) \dots
\end{aligned}
$$

c_0 will be newly notated as $\frac{a_0}{2}$, $c_1 + c_{-1}$ renamed to a_1, $c_1 + c_{-1}$ renamed to a_2, etc. $ic_1 - ic_{-1}$ to b_1, $ic_2 - ic_{-2}$ to b_2. Then we have

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi t}{l} + b_k \sin\frac{k\pi t}{l}\right)$$

Use (2.9),

$$
\begin{aligned}
a_k = c_k + c_{-k} &= \frac{1}{2l}\int_{-l}^{l} f(t)e^{-ik\frac{\pi}{l}t}\,\mathrm{d}t + \frac{1}{2l}\int_{-l}^{l} f(t)e^{ik\frac{\pi}{l}t}\,\mathrm{d}t \\
&= \frac{1}{2l}\int_{-l}^{l}\left[f(t)e^{-ik\frac{\pi}{l}t} + f(t)e^{ik\frac{\pi}{l}t}\right]\mathrm{d}t \\
&= \frac{1}{2l}\int_{-l}^{l}\left[f(t)\left(e^{-ik\frac{\pi}{l}t} + e^{ik\frac{\pi}{l}t}\right)\right]\mathrm{d}t \\
&= \frac{1}{l}\int_{-l}^{l} f(t)\cos\left(k\frac{\pi}{l}t\right)\mathrm{d}t
\end{aligned}
$$

Last equation uses the fact that,

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}).$$

And,

$$
\begin{aligned}
b_k = i(c_k - c_{-k}) &= \frac{1}{2l}\int_{-l}^{l} f(t)ie^{-ik\frac{\pi}{l}t}\,\mathrm{d}t - \frac{1}{2l}\int_{-l}^{l} f(t)ie^{ik\frac{\pi}{l}t}\,\mathrm{d}t \\
&= \frac{1}{2l}\int_{-l}^{l}\left[f(t)ie^{-ik\frac{\pi}{l}t} - f(t)ie^{ik\frac{\pi}{l}t}\right]\mathrm{d}t \\
&= \frac{i}{l}\frac{1}{2i}\int_{-l}^{l}\left[f(t)i\left(e^{-ik\frac{\pi}{l}t} - e^{ik\frac{\pi}{l}t}\right)\right]\mathrm{d}t \\
&= \frac{1}{l}\frac{1}{2i}\int_{-l}^{l}\left[f(t)\left(e^{ik\frac{\pi}{l}t} - e^{-ik\frac{\pi}{l}t}\right)\right]\mathrm{d}t \\
&= \frac{1}{l}\int_{-l}^{l} f(t)\sin\left(k\frac{\pi}{l}t\right)\mathrm{d}t
\end{aligned}
$$

Last equation uses the fact that,

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

2.4 Fourier Transform

Here we are, finally to the hardcore. All knowledges we presented before are just tools we are about employ here. Basically, the idea of Fourier series is that you have a $f(x)$ which is period function, you substitute it into formulas to calculate either for c_k in complex exponential version or for a_0, a_k and b_k in trigonometric version. And they are perfectly interchangeably, nothing mysterious, you need more practice to be familiar with them. But since we never stop propagating that Fourier analysis is an extremely powerful weapon in your mathematics arsenal, it should be extensively made use of rather than dealing with periodic function alone. Right, of course we have a knack for it.

2.4.1 Periodic Extension

As a matter of fact, I would not even want to call this a trick, preferably it is new perspective how you view functions. The figure shows function $f(x)$

Reserve a picture for period extension1

Figure 2.1: period extension1

Reserve a picture for period extension2

Figure 2.2: period extension2

with domain $(0, l)$, if we take l as its period, we can extend throughout the whole axis, as the second graph shows. Unfortuantly this period extension is useless, or most of period extensions are useless, because we only interested in interval $(0, l)$, who do we even bother to extend periods over the whole axis, that changes nothing at all.

There are two useful period extension we need to study, the first one is **odd period extension**, the second is **even period extension**.

Given any $f(x)$ with domain $(0, l)$, odd period extension is defined by conditions,

1. $F^o(x) = f(x)$ for $0 < x < l$

2. $F^o(x) = -f(x)$ for $-l < x < 0$

3. $F^o(x)$ has period $2l$

And even period extension is defined by conditions,

1. $F^e = f(x)$ for $0 < x < l$

2. $F^e = f(x)$ for $-l < x < 0$

3. F^e has period $2l$

Now recall and ask yourself, what kind of function would it be if you plus an odd function with an even one? Neither. Check theorem 2.2. We did not say anything about addition of odd and even fuctions, because result is neither of them. Take a look at trigonometric version of Fourier series again,

$$f(x) = a_0 + \sum_{k=0}^{\infty}(a_k \cos kt + b_k \sin kt)$$

Surprised, right? If we did not mention this you won't notice that Fourier series is a non-odd-or-even function, then F^e has a cosine expansion and F^o has a sine expansion, it has to be like this, because we are unable to express an even(odd) funtion with a function which is summation of addition of odd and even functions.

Although it is now a periodic function throughout the whole axis, we merely interested in interval $(0, l)$. We perform Fourier expansion on this interval,

$$\begin{aligned} f(x) = F^e(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{l}\right) \\ &= F^o(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{l}\right) \end{aligned}$$

Notice that we use $\frac{a_0}{2}$ rather than a_0, because it is more mathematically tractable, we can write $\frac{a_0'}{2} = a_0$ to differentiate from a_0 before, but we do not need a_0 any more from now on, why don't we just take $\frac{a_0}{2}$ as our new a_0. Then follow the formula Fourier 2, Fourier 3, and Fourier 4,

$$\begin{aligned} \frac{a_0}{2} &= \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x = \frac{1}{l}\int_0^l F^e\,\mathrm{d}x \\ a_k &= \frac{2}{l}\int_0^l f(x)\cos\left(\frac{k\pi x}{l}\right)\mathrm{d}x = \frac{2}{l}\int_0^l F^e(x)\cos\left(\frac{k\pi x}{l}\right)\mathrm{d}x \\ b_k &= \frac{2}{l}\int_0^l f(x)\sin\left(\frac{k\pi x}{l}\right)\mathrm{d}x = \frac{2}{l}\int_0^l F^o(x)\sin\left(\frac{k\pi x}{l}\right)\mathrm{d}x \end{aligned}$$

Notice we make use of property of even function of integral 2.2.1, both $F^e(x)\cos\left(\frac{k\pi x}{l}\right)$ and $F^o(x)\sin\left(\frac{k\pi x}{l}\right)$ are even functions due to the facts of multiplication and addition properties 2.2. So the basic idea here it to make use of both odd period and even period extensions use calculate its Fourier expansion.

A Simple Example

We choose a simplest example here, $f(x)$ is defined over $0 < x < \pi$, $f(x) = 1$ for all $0 < x < \pi$. Its even period expansion and odd period expansion graphs are shown below,

Both have period 2π, so

$$F^o(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{l}\right) = \sum_{k=1}^{\infty} b_k \sin kx,$$

Reserve a picture for Odd $f(x)$

Figure 2.3: Odd $f(x)$

Reserve a picture for Even $f(x)$

Figure 2.4: Even $f(x)$

compute b_k,

$$\begin{aligned} b_k &= \frac{2}{l}\int_0^l F^o(x)\sin(\frac{k\pi x}{l})\,\mathrm{d}x = \frac{2}{l}\int_0^l 1\sin(\frac{k\pi x}{l})\,\mathrm{d}x \\ &= \frac{2}{\pi}\Big[-\frac{1}{k}\cos(kx)\Big]\Big|_0^\pi \\ &= \frac{2}{\pi}\Big[-\frac{1}{k}\cos(k\pi)+\frac{1}{k}\cos 0\Big] \end{aligned}$$

if k is odd,

$$\begin{aligned} &= \frac{2}{\pi}\Big(\frac{1}{k}+\frac{1}{k}\Big) \\ &= \frac{2}{\pi}\frac{2}{k} \\ &= \frac{4}{k\pi} \end{aligned}$$

or if k is even

$$\begin{aligned} &= \frac{2}{\pi}\Big(-\frac{1}{k}+\frac{1}{k}\Big) \\ &= 0 \end{aligned}$$

Odd period expansion is

$$F^o(x) = \sum_{k=1+2t}^{\infty}\frac{4}{k\pi}\sin(kx) \qquad t \text{ is positive integer starts from 1}$$

Turn to even expansion,

$$F^e(x) = \frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos(kx)$$

compute a_k,

$$\begin{aligned} a_k &= \frac{2}{\pi}\int_0^{\pi} F^e(x)\cos(kx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} 1\cos(kx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\left[\frac{1}{k}\sin(kx)\right]\Big|_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{1}{k}\sin(k\pi) - \frac{1}{k}\sin 0\right] \\ &= 0 \end{aligned}$$

And we also need to know $\frac{a_0}{2}$,

$$\frac{a_0}{2} = \frac{1}{\pi}\int_0^{\pi} \mathrm{d}x = \frac{1}{\pi}[x]|_0^{\pi} = 1$$

Then even expansion is

$$F^e(x) = \frac{a_0}{2} = 1$$

2.4.2 Fourier Transform

From now on, we will stick to exponential verison of Fourier series which will save us considerable trouble in writing and computation. Recall the most important thing, Fourier series of period $2l$, we have learned by far,

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{l}t} \qquad c_k = \frac{1}{2l}\int_{-l}^{l} f(t)e^{-ik\frac{\pi}{l}t}\,\mathrm{d}t$$

From now on, we no longer impose the restriction of period functions, you will see later that Fourier analysis deals with function from a broader range than what we are doing here.

For heuristic reasons, we will use a 'special' function here,

$$f(t) = 0, \qquad |t| > L,\ L \text{ is any arbitrary positve number}$$

It might be strange why we want a function looks like this, it will be clear soon. We can get a Fourier expansion for part of $f(t)$ with $-L < t < L$ by using periodic extension. Define $F_L(t) = f(t)$ for $-L < t \leq L$ and $F_L(t)$ has period of $2L$. Naturally, we have

$$F_L = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{L}t} \qquad c_k = \frac{1}{2L}\int_{-L}^{L} f(t)e^{-ik\frac{\pi}{L}t}\,\mathrm{d}t \tag{2.10}$$

Essentially, we have not changed anything critical. The idea is to make the original function a segment (a full period) of whole periodic funtion, then we can use Fourier tools to analyse it.

If we define the k^{th} frequency to be $\omega_k = k\frac{\pi}{L}$, then calculate the first difference,

$$\omega_{k+1} - \omega_k = (k+1)\frac{\pi}{L} - k\frac{\pi}{L} = \frac{\pi}{L} = \Delta\omega \tag{2.11}$$

Conventionally, we also define

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t 3} \tag{2.12}$$

Look at (2.10), what are the differences between the equation above and (2.10)? If you have done your calculus well in your first year math, you would immediately tell that,

$$c_k = \frac{1}{2L}\int_{-L}^{L} f(t)e^{-ik\frac{\pi}{L}t}\,\mathrm{d}t = \frac{1}{2L}\int_{-\infty}^{\infty} f(t)e^{-ik\frac{\pi}{L}t}\,\mathrm{d}t$$

Because $f(t) = 0$ for all $|t| \geq L$, then there is no difference whether we set interval as $[-L, L]$ or $[-\infty, \infty]$. Next, use $\omega_k = k\frac{\pi}{L}$, yields

$$c_k = \frac{1}{2L}\int_{-\infty}^{\infty} f(t)e^{-i\omega_k t}\,\mathrm{d}t = \frac{1}{2L}\hat{f}(\omega_k)$$

This isn't over yet, try not to be lost here. We want to put $\Delta\omega$ into last equation by using $L = \frac{\pi}{\Delta\omega}$ from (2.11),

$$c_k = \frac{1}{2L}\hat{f}(\omega_k) = \frac{1}{2\pi}\hat{f}(\omega_k)\Delta\omega$$

[3] This is actually Fourier transform, but it is not clear why it looks like this at the moment.

What we are doing above is trying to prepare elements (such as $\Delta\omega$) for an integral. So the question remains 'what integral is it?', then watch carefully below.

Substitute $c_k = \frac{1}{2\pi}\hat{f}(\omega)\Delta\omega$ back to F_L in (2.10),

$$f(t) = F_L(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi}\hat{f}(\omega_k)\Delta\omega e^{ik\frac{\pi}{L}t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\hat{f}(\omega)e^{i\omega_k t}\Delta\omega$$

As your basic integral knowledges tell you,

$$\frac{1}{2\pi}\lim_{\Delta\omega\to 0}\sum_{k=-\infty}^{\infty}\hat{f}(\omega)e^{i\omega_k t}\Delta\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}\,\mathrm{d}\omega \tag{2.13}$$

A notice has to be made here, we are actually letting $L \to \infty$, because from (2.11) $\frac{\pi}{L} = \Delta\omega$, results $\Delta\omega \to 0$.

Together with (2.12) and (2.13), we have Fourier transform and its inverse Fourier transform,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}\,\mathrm{d}\omega \tag{2.14}$$

These are Fourier transform pairs, conventionally we denote $\hat{f}(\omega)$ as the Fourier transform of $f(t)$. A function $f(t)$ can go back and forth from its time domain to frequency domain by Fourier transform and its inverse operation, only if the function is continuous and integrable. Some textbook use $\mathcal{F}[f(t)] = \hat{f}(\omega)$ as Fourier transform operator, which we seldom need it.

2.4.3 Properties of the Fourier Transform

This is the most important part of Fourier theory, but nothing really difficult here, you just need to get familiar with notations and operations. We will present some examples to further your understanding.

Linearity

Let α and β be any constants, $f(t)$ and $g(t)$ two continuous functions, and a linear combination $h(t) = \alpha f(t) + \beta g(t)$, the Fourier transform of $h(t)$ is

$$\begin{aligned}
\hat{h}(\omega) &= \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} [\alpha f(t) + \beta g(t)] e^{-i\omega t}\,\mathrm{d}t \\
&= \alpha \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\,\mathrm{d}t + \beta \int_{-\infty}^{\infty} g(t) e^{-i\omega t}\,\mathrm{d}t \\
&= \alpha \hat{f}(\omega) + \beta \hat{g}(\omega)
\end{aligned}$$

Simply speaking, linearity of Fourier transform is completely due to the linearity of integration rules.

Time Shifting

Reserve a picture for time shifting

Figure 2.5: Time shifting

Suppose we have a function $h(t) = f(t - t_0)$, if $t_0 > 0$, use your high school math, it is clear that $h(t)$ is to the right of $f(t)$ as figure shows.

Fourier transform of $h(t)$ is

$$\begin{aligned}
\hat{h}(\omega) &= \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(t - t_0) e^{-i\omega t}\,\mathrm{d}t
\end{aligned}$$

If we denote $t' = t - t_0$,

$$= \int_{-\infty}^{\infty} f(t') e^{-i\omega (t' + t_0)}\,\mathrm{d}t$$

Also we calculate the differential, $\mathrm{d}t = \mathrm{d}(t' - t_0) = \mathrm{d}t'$,

$$= \int_{-\infty}^{\infty} f(t')e^{-i\omega(t'+t_0)}\,\mathrm{d}t'$$
$$= \int_{-\infty}^{\infty} f(t')e^{-i\omega t'}e^{-i\omega t_0}\,\mathrm{d}t'$$

$e^{-i\omega t_0}$ is constant,

$$= e^{-i\omega t_0}\int_{-\infty}^{\infty} f(t')e^{-i\omega t'}\,\mathrm{d}t'$$
$$= e^{-i\omega t_0}\hat{f}(\omega)$$

The last equation holds because Fourier transform changes the function regardless of any argument.

Scaling

If we build a new funtion,

$$h(t) = f\left(\frac{t}{\alpha}\right), \text{ by a scaling factor } \alpha > 0.$$

Fourier transform is

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\,\mathrm{d}t$$
$$= \int_{-\infty}^{\infty} f\left(\frac{t}{\alpha}\right)e^{-i\omega t}\,\mathrm{d}t$$

We denote $t = \alpha t'$

$$= \int_{-\infty}^{\infty} f(t')e^{-i\omega\alpha t'}\,\mathrm{d}t$$

We calculate the differential, $\mathrm{d}t = \mathrm{d}(\alpha t') = \alpha\mathrm{d}t'$

$$= \alpha\int_{-\infty}^{\infty} f(t')e^{-i\omega\alpha t'}\,\mathrm{d}t'$$
$$= \alpha\hat{f}(\alpha\omega)$$

The last equation holds because we view $\alpha\omega$ as the new frequency (don't forget that α is a scaling factor).

Differentiation

Let $h(t) = f'(t)$, then Fourier transform is

$$\begin{aligned}\hat{h}(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} f'(t)e^{-i\omega t}\,\mathrm{d}t\end{aligned}$$

Integrate by parts, $u = e^{-i\omega t}$, so $\mathrm{d}u = -i\omega e^{-i\omega t}\mathrm{d}t$. Also $v = f(t)$, then $\mathrm{d}v = f'(t)\mathrm{d}t$.

$$\begin{aligned}&= \int_{-\infty}^{\infty} u\mathrm{d}v \\ &= uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\mathrm{d}u \\ &= e^{-i\omega t}f(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(-i\omega)e^{-i\omega t}\mathrm{d}t\end{aligned}$$

Assume that $f(\pm\infty) = 0$

$$\begin{aligned}&= -\int_{-\infty}^{\infty} f(t)(-i\omega)e^{-i\omega t}\mathrm{d}t \\ &= i\omega\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t \\ &= i\omega\hat{f}(\omega)\end{aligned}$$

Parseval's Relation

We define the energy of signal $f(t)$ is

$$\int_{-\infty}^{\infty} |f(t)|^2\mathrm{d}t = \int_{-\infty}^{\infty} f(t)\overline{f(t)}\mathrm{d}t$$

$\overline{f(t)}$ is the conjugate of $f(t)$,

$$\overline{f(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)e^{i\omega t}}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)}e^{-i\omega t}\mathrm{d}\omega$$

Substitute back into last equation,

$$\int_{-\infty}^{\infty} f(t)\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)}e^{-i\omega t}\mathrm{d}\omega\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\overline{\hat{f}(\omega)}e^{-i\omega t}\mathrm{d}\omega\mathrm{d}t$$

Iterative integration,

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)}\left[\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t\right]\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(\omega)}\hat{f}(\omega)\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\mathrm{d}\omega
\end{aligned}
$$

What we get is the formula of **Parseval's relation**,

$$\int_{-\infty}^{\infty} |f(t)|^2\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\mathrm{d}\omega$$

$|\hat{f}(\omega)|^2$ is called **power spectrum** of $f(t)$, then $|\hat{f}(\omega)|$ is **Fourier spectrum**.

Duality

If we have a $\hat{f}(\omega)$, we exchange the variable of time and frequency, and set $\hat{f}(t) = g(t)$. The Fourier transform of $g(t)$ is,

$$
\begin{aligned}
\hat{g}(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \hat{f}(t)e^{-i\omega t}\,\mathrm{d}t
\end{aligned}
$$

Change notation, $t = s$

$$= \int_{-\infty}^{\infty} \hat{f}(s)e^{-i\omega s}\,\mathrm{d}s$$

Inverse Fourier transform of $f(t)$ is,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,\mathrm{d}\omega$$

Change notation, $\omega = s$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)e^{ist}\,\mathrm{d}s$$

Together with

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} \hat{f}(s) e^{-i\omega s} \, \mathrm{d}s$$

We find that,

$$\hat{g}(\omega) = 2\pi f(-\omega)$$

Convolution

A *filter* is described by a funtion of $\hat{H}(\omega)$, for instance, $\hat{H}(\omega) = 1$ for desirable frequency and $\hat{H}(\omega) = 0$ for undesirable frequency. If a signal $f(t)$ is fed into the filter, an output is $g(t)$, its Fourier transform, we set as,

$$\hat{f}(\omega)\hat{H}(\omega) = \hat{g}(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} \, \mathrm{d}t$$

Inverse Fourier transform,

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{H}(\omega) e^{i\omega t} \, \mathrm{d}\omega$$

$\hat{f}(\omega)$ is easy to get, it is just a Fourier transform of $f(t)$, but we need to change notation,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} \, \mathrm{d}\tau$$

And substitute into $g(t)$,

$$\begin{aligned} g(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) e^{-i\omega\tau} \, \mathrm{d}\tau \hat{H}(\omega) e^{i\omega t} \, \mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} f(\tau) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{H}(\omega) e^{i\omega(t-\tau)} \, \mathrm{d}\omega \right] \mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} f(\tau) H(t-\tau) \, \mathrm{d}\tau \end{aligned}$$

The last equation is called **convolution**, formally denoted

$$(f * H)(t) = \int_{-\infty}^{\infty} f(\tau) H(t-\tau) \, \mathrm{d}\tau \tag{2.15}$$

The above relationship can be summarized,

$$\hat{g}(\omega) = \hat{f}(\omega)\hat{H}(\omega)$$
$$g(t) = (f * H)(t)$$

2.4.4 Examples

After studying so many properties[4], we really to study through some examples to make sure we comprehend them.

Example 1 We have a signal turned on at $t = 0$ then decays exponentially,

$$f(t) = \begin{cases} e^{-at} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

from some $a > 0$. The Fourier transform of this signal is

$$\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t = \int_{-\infty}^{\infty} e^{-at}e^{-i\omega t}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} e^{-t(a+i\omega)}\,\mathrm{d}t \\ &= \left.\frac{e^{-t(a+i\omega)}}{-(a+i\omega)}\right|_0^{\infty} \\ &= -\frac{e^{-0(a+i\omega)}}{-(a+i\omega)} \\ &= \frac{1}{a+i\omega} \end{aligned}$$

Example 2 We have boxcar normalized function as below, signal turned

Reserve a picture for impulse generalization

Figure 2.6: Impulse generalization

[4] Actually these are only a portion of its all properties.

on at $t = -\frac{1}{2}$ and off at $t = \frac{1}{2}$, impulse height is 1. Denote

$$\text{rect}(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

'rect' means rectangle. The Fourier transform of the signal is

$$\begin{aligned} \widehat{\text{rect}}(\omega) &= \int_{-\infty}^{\infty} \text{rect}(t) e^{-i\omega t} \, \mathrm{d}t \\ &= \int_{-1/2}^{1/2} e^{-i\omega t} \, \mathrm{d}t \\ &= \frac{e^{-i\omega t}}{-i\omega} \Big|_{-1/2}^{1/2} \\ &= \frac{e^{-i\omega/2}}{-i\omega} - \frac{e^{i\omega/2}}{-i\omega} \\ &= \frac{e^{i\omega/2} - e^{-i\omega/2}}{i\omega} \end{aligned}$$

We can stop here, or we use Euler's formula,

$$\begin{aligned} &= \frac{1}{i\omega}\left(\cos\frac{\omega}{2} + i\sin\frac{\omega}{2} - \cos\frac{\omega}{2} + i\sin\frac{\omega}{2} \right) \\ &= \frac{1}{i\omega} 2i \sin\frac{\omega}{2} \\ &= \frac{2}{\omega}\sin\frac{\omega}{2} \end{aligned}$$

There is a function named 'sinc', $\text{sinc}\,\omega = \frac{\sin\omega}{\omega}$. Then $\widehat{\text{rect}}(\omega) = \text{sinc}\,\frac{\omega}{2}$.

Example 3 We have a rectangular pulse of height H, width W and centre C.

Reserve a picture for rectangular impulse

Figure 2.7: rectangular impulse

We can denote this rectangular impulse function,

$$r(t) = H\text{rect}\left(\frac{t-C}{W}\right)$$

from previous knowledge of shifting and scaling, we shift *rect*[5] by C to the right, and scale by a factor of W. The signal gets turned on at $C - \frac{W}{2}$ and off at $C + \frac{W}{2}$. This example is just a generalization of last one, study it carefully, it gives a general method to cope with rectangular impulse functions. We build the method in three steps.

The *first step*, we consider the scaling alone, assuming no shifting $C = 0$. Then we denote

$$f(t) = \text{rect}(t), \qquad h(t) = r_1(t) = \text{rect}\left(\frac{t}{W}\right) = f\left(\frac{t}{W}\right)$$

Its Fourier transform according to scaling property is,

$$\hat{h}(\omega) = \hat{r}_1(\omega) = W\widehat{\text{rect}}(W\omega) \tag{2.16}$$

Use sinc function,

$$\widehat{\text{rect}}(W\omega) = \text{sinc}\,\frac{W\omega}{2} = \frac{\sin\frac{W\omega}{2}}{\frac{W\omega}{2}} = \frac{2}{W\omega}\sin\frac{W\omega}{2}$$

Thus,

$$\hat{h}(\omega) = \hat{r}_1(\omega) = \frac{2}{\omega}\sin\frac{W\omega}{2}$$

The *second step*, we allow a nonzero C, denote

$$f(t) = r_1(h), \qquad h(t) = r_2(t) = \text{rect}\left(\frac{t-C}{W}\right) = r_1(t-C)$$

According to property of time shifting, its Fourier transform is

$$\hat{h}(\omega) = \hat{r}_2(\omega) = e^{-i\omega C}\hat{f}(\omega) = e^{-i\omega C}\hat{r}_1(\omega)$$

Use equation (2.16),

$$\hat{h}(\omega) = \hat{r}_2(\omega) = e^{-i\omega C}W\widehat{\text{rect}}(W\omega)$$

[5] rect is always a normalized boxcar impulse function without other explicit emphasizing.

The *last step*, use linearity property, we denote

$$\hat{r}(\omega) = H\hat{r}_2(\omega) = He^{-i\omega C}W\widehat{\text{rect}}(W\omega)$$

Use sinc function again,

$$\begin{aligned} He^{-i\omega C}W\widehat{\text{rect}}(W\omega) &= He^{-i\omega C}W \operatorname{sinc}\frac{W\omega}{2} \\ &= He^{-i\omega C}W\frac{\sin\frac{W\omega}{2}}{\frac{W\omega}{2}} \\ \hat{r}(\omega) &= \frac{2H}{\omega}e^{-i\omega C}\sin\frac{W\omega}{2} \end{aligned} \tag{2.17}$$

Notation might look a little bit messy, but the idea is very simple that we don't need to perform a Fourier transform directly on rectangular impulse function, we can decompose the steps into three, each step uses the result of last step's Fourier transform. Don't get lost at notation, focus on the essence.

Example 4 Here is another generalized example 2.8 of rectangular impulse functions,

Reserve a picture for several rectangular impulses

Figure 2.8: several rectangular impulses

$$s_n(t) = r(t) \text{ with } \begin{cases} H = 2, \quad W = 1, \quad C = -1.5 & \text{for n=1} \\ H = 1, \quad W = 2, \quad C = 1 & \text{for n=2} \\ H = 0.5, \quad W = 2, \quad C = 3 & \text{for n=3} \end{cases}$$

We define $s(t) = s_1(t) + s_2(t) + s_3(t)$, use result from last example (2.17),

$$s(t) = \frac{4}{\omega}e^{\frac{3}{2}i\omega}\sin\frac{\omega}{2} + \frac{2}{\omega}e^{-i\omega}\sin\omega + \frac{1}{\omega}e^{-3i\omega}\sin\omega$$

Example 5 Suppose we use filter

$$\hat{H}(\omega) = \frac{2}{\omega}\sin\frac{\omega}{2}.$$

To see it in time domain,

$$H(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Apply the filter to a signal $f(t)$, output at time t is,

$$(f * H)(t) = \int_{-\infty}^{\infty} f(t-\tau)H(\tau)\,\mathrm{d}\tau = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t-\tau)\,\mathrm{d}\tau$$

Define $\tau' = t - \tau$, then $\mathrm{d}\tau' = -\mathrm{d}\tau$, thus

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t-\tau)\,\mathrm{d}\tau = -\int_{t+\frac{1}{2}}^{t-\frac{1}{2}} f(\tau')\mathrm{d}\tau' = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} f(\tau')\mathrm{d}\tau'$$

Notice that change the variable will also change the interval.

Some graphics will be added here.

2.5 Discrete Fourier Transform

Continuous Fourier transform has great value in analytical study of signal, but in real world we can only deal with discrete signals in computers. Notation will be different, we use $x[n]$ for discrete-time signal, and $x(t)$ for continuous-time signal. We will start from periodic signals to nonperiodic signals just as we did in continuous case.

2.5.1 Periodic Signals

First we have a conintuous signal $x(t)$ which has period $2l$, we use machine to measure the signal, of course we cannot measure every instant (if we can, there is no need to study discrete signals any more). So unfortuantely we cannot use

$$c_k = \frac{1}{2l}\int_0^{2l} x(t)e^{-ik\frac{\pi}{l}t}\,\mathrm{d}t, \tag{2.18}$$

because we need to know $x(t)$ for every possible t. But we can approximate this integral by a Riemann sum.

Suppose we measure $x(t)$ at equally spaced value of t for N times. We pick t at

$$t = \frac{2l}{N}, 2\frac{2l}{N}, 3\frac{2l}{N}, \dots, N\frac{2l}{N}.$$

For t in the any interval between two measure points,

Reserve a picture for discrete signal Riemann sum

Figure 2.9: discrete signal Riemann sum

$$n\frac{2l}{N} \leq t \leq (n+1)\frac{2l}{N}$$

Notice the integral (2.18) above, $x(t)e^{-ik\frac{\pi}{l}t}$ is actually a function of t, we can feed it all measure points to calculate function values. Say we pick a measure point $t = n\frac{2l}{N}$,

$$x(n\frac{2l}{N})e^{-ik\frac{\pi}{l}n\frac{2l}{N}} = x(n\frac{2l}{N})e^{-ik\pi n\frac{2}{N}} = x(n\frac{2l}{N})e^{-2\pi i\frac{kn}{N}}$$

Then we approximate the integral over $n\frac{2l}{N} \leq t \leq (n+1)\frac{2l}{N}$, this is just the segment area of whole integral. We approximate this area by using area of a rectangle under it, which is

$$x(n\frac{2l}{N})e^{-2\pi i\frac{kn}{N}}\frac{2l}{N}$$

$\frac{2l}{N}$ is the width of the rectangle. Finally we can use a Riemann sum to approximate the integral, then

$$c_k \approx \frac{1}{2l}\sum_{n=1}^{N} x(n\frac{2l}{N})e^{-2\pi i\frac{kn}{N}}\frac{2l}{N} = \frac{1}{N}\sum_{n=1}^{N} x(n\frac{2l}{N})e^{-2\pi i\frac{kn}{N}}$$

We will change notation slightly in order to be symmetric to continuous Fourier transform, set $c_k \approx \hat{x}[k]$, and $x[n] = x(n\frac{2l}{N})$. Then

$$\hat{x}[k] = \frac{1}{N}\sum_{n=1}^{N} x[n]e^{-2\pi i\frac{kn}{N}} \tag{2.19}$$

This is what we called *discrete Fourier transform*, quite unimpressive, right?

Of course it has a mirror side just as continuous case does, *inverse discrete Fourier transform*,

$$x[n] = \sum_{k=1}^{N} \hat{x}[k] e^{2\pi i \frac{nk}{N}} \tag{2.20}$$

Although we won't show how it is derived, but we can show why it holds. Do you still remember the when we studied the continuous Fourier transform, we define $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t$, we do not really show how it is derived, but simply defined it and derive the other one. So here we are following the same logic pattern work through them.

Now we will show when (2.19) holds (2.20) must be true. We substitute $\hat{x}[k]$ into (2.20),

$$x[n] = \sum_{k=1}^{N} e^{2\pi i \frac{nk}{N}} \hat{x}[k] = \sum_{k=1}^{N} e^{2\pi i \frac{nk}{N}} \frac{1}{N} \sum_{n=1}^{N} x[n] e^{-2\pi i \frac{kn}{N}}$$

However, we need to rename the variable, because n in the rightmost function is different from n in the leftmost term.

$$\begin{aligned}
x[n] &= \sum_{k=1}^{N} e^{2\pi i \frac{nk}{N}} \frac{1}{N} \sum_{n'=1}^{N} x[n'] e^{-2\pi i \frac{kn'}{N}} \\
&= \frac{1}{N} \sum_{n'=1}^{N} \sum_{k=1}^{N} e^{2\pi i \frac{nk}{N}} e^{-2\pi i \frac{kn'}{N}} x[n'] \\
&= \frac{1}{N} \sum_{n'=1}^{N} \sum_{k=1}^{N} e^{2\pi i \frac{k(n-n')}{N}} x[n'] \\
&= \frac{1}{N} \sum_{n'=1}^{N} x[n'] \sum_{k=1}^{N} e^{2\pi i \frac{k(n-n')}{N}} \\
&= \frac{1}{N} \sum_{n'=1}^{N} x[n'] \boxed{\sum_{k=1}^{N} \left(e^{2\pi i \frac{(n-n')}{N}} \right)^k}
\end{aligned}$$

We need to study a little bit about the term in the box. If $n' = n$, then

$$\sum_{k=1}^{N} \left(e^{2\pi i \frac{(n-n')}{N}} \right)^k = \sum_{k=1}^{N} e^0 = \sum_{k=1}^{N} 1 = N$$

2.5.2 Properties of Discret Fourier Transform

These properties are similar to those in continuous Fourier transform.

Time shifting

Let $x[n]$ be a discrete-time signal of period N[6]. n_0 is any integer, we define $y[n] = x[n - n_0]$, and its discrete Fourier transform is

$$\begin{aligned}\hat{y}[k] &= \frac{1}{N} \sum_{n=1}^{N} y[n] e^{-2\pi i \frac{kn}{N}} \\ &= \frac{1}{N} \sum_{n=1}^{N} x[n - n_0] e^{-2\pi i \frac{kn}{N}}\end{aligned}$$

Substitute $m = n - n_0$,

$$\begin{aligned}&= \frac{1}{N} \sum_{m+n_0=1}^{N-n_0} x[m] e^{-2\pi i \frac{k(m+n_0)}{N}} \\ &= \frac{1}{N} \sum_{m=-n_0+1}^{N-n_0} x[m] e^{-2\pi i \left(\frac{km}{N} + \frac{kn_0}{N} \right)} \\ &= \frac{1}{N} \sum_{m=-n_0+1}^{N-n_0} x[m] e^{-2\pi i \frac{km}{N}} e^{-2\pi i \frac{kn_0}{N}} \\ &= e^{-2\pi i \frac{kn_0}{N}} \left(\frac{1}{N} \sum_{m=-n_0+1}^{N-n_0} x[m] e^{-2\pi i \frac{km}{N}} \right)\end{aligned}$$

Here is the idea, notice

$$\sum_{m=-n_0+1}^{N-n_0},$$

[6] It means the siginal will come back to the same value every N measured point (sampling point) after.

$N - n_0 + n_0 - 1 = N - 1$, then we can change the interval and keep the summation the same only if we keep the sampling points to be $N-1$. So

$$\sum_{m=1}^{N}$$

will also do. Thus,

$$\begin{aligned}\hat{y}[k] &= e^{-2\pi i \frac{kn_0}{N}} \left(\frac{1}{N} \sum_{m=1}^{N} x[m] e^{-2\pi i \frac{km}{N}} \right) \\ &= e^{-2\pi i \frac{kn_0}{N}} \hat{x}[k]\end{aligned}$$

Parseval's Relation

Follow the steps we did for continuous case, this is quick,

$$\begin{aligned}\sum_{k=1}^{N} |\hat{x}[k]|^2 &= \sum_{k=1}^{N} \overline{\hat{x}[k]} \hat{x}[k] \\ &= \sum_{k=1}^{N} \left(\frac{1}{N} \sum_{n=1}^{N} \overline{x[n] e^{-2\pi i \frac{kn}{N}}} \right) \hat{x}[k] \\ &= \sum_{k=1}^{N} \left(\frac{1}{N} \sum_{n=1}^{N} \overline{x[n]} e^{2\pi i \frac{kn}{N}} \right) \hat{x}[k] \\ &= \frac{1}{N} \sum_{n=1}^{N} \overline{x[n]} \left(\sum_{k=1}^{N} e^{2\pi i \frac{kn}{N}} \hat{x}[k] \right) \\ &= \frac{1}{N} \sum_{n=1}^{N} \overline{x[n]} x[n] \\ &= \frac{1}{N} \sum_{n=1}^{N} |x[n]|^2\end{aligned}$$

2.5.3 Aperiodic Signals

Same procedure will be implemented as the continuous case section 2.2. Again for simplicty, we develop a discrete function that $x[n] = 0$ for $|n| \geq \frac{N}{2}$, where N is period, and an even number. We can get a disrete Fourier transform for the part of $x[n]$ with $|n| \leq \frac{N}{2}$ by using periodic extension. So we define a function

$$x_N[n] = x[n] \text{ for } 1\frac{N}{2} < n \leq \frac{N}{2}$$
$$x_N[n] \text{ has period N}$$

Then discrete Fourier transform pair are

$$x_N[n] = \sum_{-\frac{N}{2}<k\leq\frac{N}{2}} \widehat{x_N}[k]e^{2\pi i\frac{nk}{N}} \qquad \widehat{x_N}[k] = \frac{1}{N}\sum_{-\frac{N}{2}<k\leq\frac{N}{2}} x[n]e^{-2\pi i\frac{nk}{N}} \quad (2.21)$$

We define k^{th} frequency to be $\omega_k = 2\pi\frac{k}{N}$, then

$$\omega_k - \omega_{k-1} = 2\pi\frac{k}{N} - 2\pi\frac{k-1}{N} = \frac{2\pi}{N} = \Delta\omega$$

And also define

$$\hat{x}(\omega)[k] = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n}.$$

Then,

$$\widehat{x_N}[k] = \frac{1}{N}\sum_{-\frac{N}{2}<n\leq\frac{N}{2}} x[n]e^{-2i\pi\frac{nk}{N}}$$

$x[n] = 0$ for all $|n| \geq \frac{N}{2}$, it won't make a difference whether we sum up over interval $[-\frac{N}{2}, \frac{N}{2}]$ or $(-\infty, \infty)$, since all zero terms won't count,

$$\begin{aligned}\widehat{x_N}[k] &= \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-i2\pi\frac{k}{N}n} \\ &= \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-i\omega_k n} \\ &= \frac{1}{N}\hat{x}(\omega_k)\end{aligned}$$

because $N = \frac{2\pi}{\Delta\omega}$,

$$= \frac{1}{2\pi}\hat{x}(\omega_k)\Delta\omega$$

Substitute the result above into $x[n]$,

$$x[n] = x_N[n] = \sum_{-\frac{N}{2}<k\leq\frac{N}{2}} \frac{1}{2\pi}\hat{x}(\omega_k)\Delta\omega e^{2\pi i\frac{nk}{N}} = \frac{1}{2\pi} \sum_{-\frac{N}{2}<k\leq\frac{N}{2}} \hat{x}(\omega_k)e^{2\pi i\frac{nk}{N}}\Delta\omega$$

We will modify the summation restriction like this,

$$\frac{2\pi}{N}(-\frac{N}{2} < k \leq \frac{N}{2}) = -\pi < \omega_k \leq \pi,$$

obviously you can see that we are intentionally ω in summation restriction. Thus, we have

$$\frac{1}{2\pi} \sum_{-\pi<\omega_k\leq\pi} \hat{x}(\omega_k)e^{i\omega_k n}\Delta\omega$$

But note that the argument of the summation is still k, not ω. Now we could rightfully approximate the Riemann sum to integral by letting $N \to \infty$. Then $\Delta\omega = \frac{2\pi}{N} = 0$. We conclude that

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(\omega)\,\mathrm{d}\omega \qquad \hat{x}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-i\omega n} \tag{2.22}$$

$\hat{x}(\omega)$ is the discrete Fourier transform of $x[n]$, and is given a name 'spectrum' of $x[n]$. Compare (2.22) and (2.4.2).

2.6 z-transform

2.6.1 Linear, Time Invariant Systems

Linear, time invariant (LTI) system has two version, continuous and discrete. Most of differential equation you learned are continuous LTI. The system is linear, because if we feed system with a signal $ax_1 + bx_2$, we receive a signal $ay_1 + by_2$. Time invariant means that for any time shifted signal $x(t - s)$, generates the time shifted output $y(t - s)$. As you no doubt have guessed, difference equations are discrete LTI system. We will present examples later.

2.6.2 Impulse Response Function

We have seen Dirac delta function,

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

It simply models the idea that there is a unit impulse when time hits 0. We denote $h[n]$ the response from the unit impulse at time 0, naturally it is our *impulse response function*. We need to modify Dirac delta funtion to make it more general,

$$\delta_k[n] = \begin{cases} 1 & \text{if } n = k \text{ and } k > 0 \\ 0 & \text{if } n \neq k \end{cases}$$

it becomes a unit impulse at time k, rest of instants are 0. According to time shift property of all functions, $\delta_k[n] = \delta[n-k]$, accordingly impulse response function will become $h[n-k]$ due to LTI time invariant property, because time shifted input $\delta[n-k]$ generates time shifted $h[n-k]$ output.

Say we have an input signal $x[3]$, the signal is input at time instant 3, we have a crucial observation,

$$x[3] = \cdots + a_1\delta_1[3] + a_2\delta_2[3] + a_3\delta_3[3] + a_4\delta_4[3] + a_5\delta_5[3]\cdots$$

Besides $a_3\delta_3[3] = a_3$, in general $x[k] = a_k$, rest of terms are zero. Why do we bother to write this? Because this is a linear combination,

$$x[n] = \sum_{k=-\infty}^{k=\infty} a_k\delta_k[n]$$

We have seen that the output corresponding to $\delta_k[n]$ is $h[n-k]$. So the output corresponding $x[n]$, by linearity of LTI,

$$y[n] = \sum_{k=-\infty}^{k=\infty} a_k h[n-k] = \sum_{k=-\infty}^{k=\infty} x[k]h[n-k] = (h * x)[n] \qquad (2.23)$$

This is *discrete form of convolution*, compare with (2.15).

2.6.3 z-transform

One important feature of discrete-time LTI systems is that all *exponential signals* are *eigenfunctions* for all LTI systems. It will be clear soon. If z is arbitrary complex number, then we define a signal $x[n] = z^n$, which is an complex exponential signal. Feed the signal to LTI system with impulse function $h[n]$,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = H(z)z^n$$

This is called z-**transform** of the impulse response $h[n]$. z^n is eigenfunction and $H(z)$ is eigenvalue, where

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}.$$

Where z^n is feed into a LTI system, result is just z^n multiply a constant, $H(z)$, which is independent of time n.

Chapter 3

Linear Filters

This chapter has close connection with time series analysis, we won't dive too deep, but neither too superficial.

3.1 Spectral density

Time series textbook prefer to using a different notation of Fourier transform from we have derived in last chapter,

$$\hat{x}(\omega) = \sum_{t=-\infty}^{\infty} x_t e^{-i\omega t}, \qquad x_t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}(\omega) e^{i\omega t}\,\mathrm{d}\omega \tag{3.1}$$

x_t is the time series we are interested, it naturally is a discrete sequence.

To put simply, spectral density is the Fourier transform of the autocovariance function. Let γ_j denote autocovariance between y_t and y_{t+j} (or y_{t-j}), which is a zero mean, stationary process,

$$\gamma_j = E(y_t - \mu_t)(y_{t+j} - \mu_{t+j}), \qquad E[y_t] = \mu_t$$

Take Fourier transform[1], we get the *power spectral density*,

$$s(\omega) = \hat{x}(\omega) = \sum_{j=-\infty}^{\infty} \gamma_j e^{-i\omega j} \tag{3.2}$$

[1] Some textbooks call this Wiener-Khinchine relation.

Because γ_j is symmetric due to its stationarity, we can expand the series,

$$\begin{aligned} s(\omega) &= \ldots + \gamma_{-2}e^{2i\omega} + \gamma_{-1}e^{i\omega} + \gamma_0 + \gamma_1 e^{-i\omega} + \gamma_2 e^{-2i\omega} + \ldots \\ &= \gamma_0 + 2\sum_{j=1}^{\infty} \gamma_j e^{-i\omega j} \\ &= \gamma_0 + 2\sum_{j=1}^{\infty} \gamma_j [\cos(\omega j) - i \sin(\omega j)] \\ &= \gamma_0 + 2\sum_{j=1}^{\infty} \gamma_j \cos(\omega j) \end{aligned}$$

Then take inverse Fourier transform to recover γ_j

$$\gamma_j = \frac{1}{2\pi} \int_{-\infty}^{\infty} s(\omega) e^{i\omega j} \, \mathrm{d}\omega$$

A special case, when $j = 0$,

$$\gamma_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} s(\omega) \, \mathrm{d}\omega = \mathrm{Var}(y_t)$$

since $\gamma_0 = E(y_t - \mu_t)(y_t - \mu_t) = \mathrm{Var}(y_t)$. This is variance decomposition in terms of uncorrelated components at each frequency ω.

3.2 Cramér Representations

Recall that the standard Fourier series is given by

$$f(t) = a_0 + \sum_{k=-\infty}^{\infty} [a_k \cos(kt) + b_k \sin(kt)]$$

There is an integral version of Fourier series,

$$y_t = \mu_0 + \int_{-\infty}^{\infty} \left[\alpha(\omega)\cos(\omega t) + \beta(\omega)\sin(\omega t)\right] \mathrm{d}\omega$$

which is also named **Cramér representation**, or **spectral representation**. $\alpha(\omega)$ and $\beta(\omega)$ are uncorrelated zero-mean random variables, and ω measures period. It is not necessary to consider the whole range $[-\infty, \infty]$,

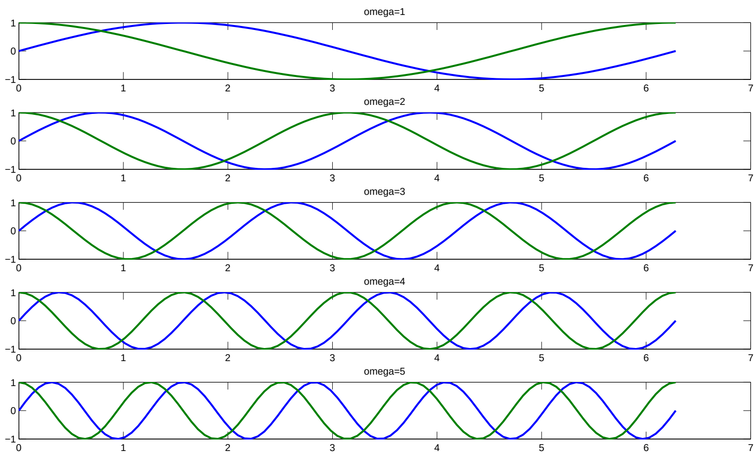

Figure 3.1: Cosine and sine with different periodicity

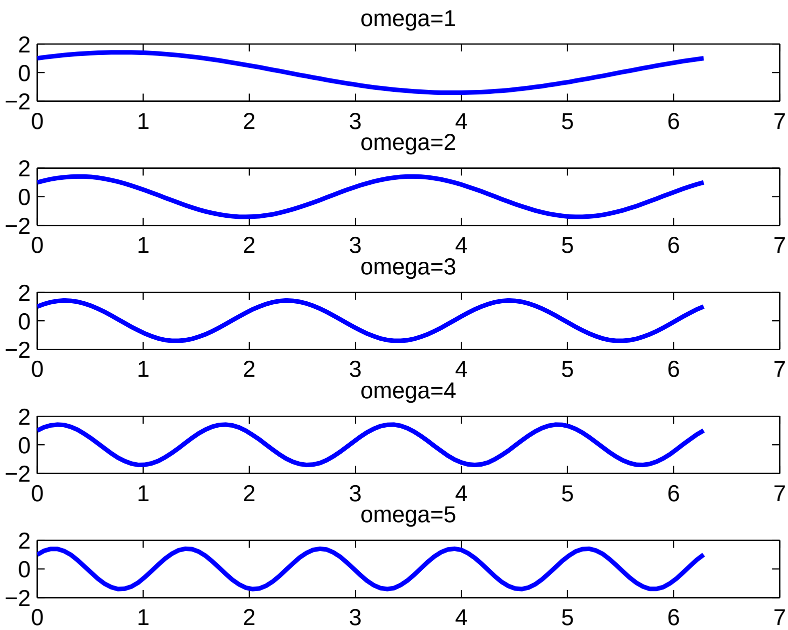

Figure 3.2: The graph of equation (3.3)

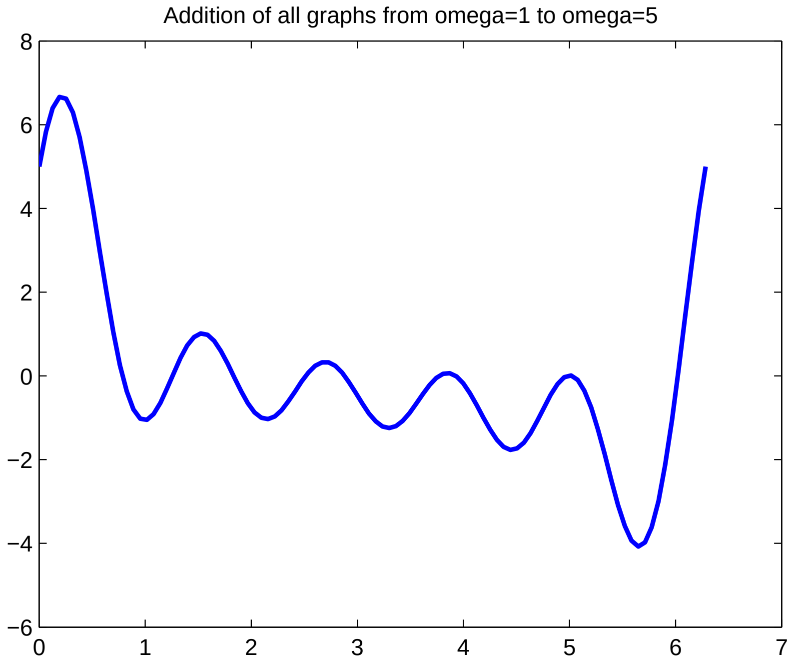

Figure 3.3: Addition of previous five graphs

and $[0, \pi]$ will suffice. The figure 3.1 shows the different period ω from 1 to 5, namely, how many circles complete within $[0, 2\pi]$. If we define

$$y_t^\omega = \mu_0 + \alpha(\omega)\cos(\omega t) + \beta(\omega)\sin(\omega t) \tag{3.3}$$

Any time series y_t can be represented by the integration of y_t^ω out of ω by choosing appropriate $\alpha(\omega)$, $\beta(\omega)$ and μ_0

$$y_t = \int_0^\pi \alpha(\omega)\cos(\omega t)\,\mathrm{d}\omega + \int_0^\pi \beta(\omega)\sin(\omega t)\,\mathrm{d}\omega$$

If we set $\mu_0 = 0$, $\alpha(\omega) = \beta(\omega) = 1$, the graph of y_t^ω is showed in figure 3.2 We can pretend we are integrating y_t^ω out of ω by adding these five graphs together, see figure 3.3. To give economic meaning for ω, we define

$$p = \frac{2\pi}{\omega} \tag{3.4}$$

as the the number of units of time to complete one cycle. Then if we know the time unit p, we can figure out ω by using equation (3.4). We usually set business cycle between 6 and 40 quarters, then we have an interval for $\omega \in [2\pi/6, 2\pi/40]$, i.e. $[0.157, 1.047]$.

3.3 Linear Filters

Filtering is a engineering concept, most famous in electronic engineering. The general form of linear filter is a linear combination of discrete-time signal y_t (in macroeconomics, the signal is time series),

$$y_t^f = C(L)y_t = \sum_j c_j y_{t-j}$$

where $C(L)$ is

$$C(L) = \cdots + c_{-1}L^{-1} + c_0 + c_1 L + \cdots$$

The weights sequence $C(L)$ can be either finite or infinite, finite sequence is a moving average filter.

The filter is most understandable when you look at your media play on your computer, open the equaliser setting panel you will see 3.4, The leftmost three scrolls control low frequencies from 32 Hz to 125 Hz, while high frequencies starts from 4 KHz to 16 KHz on the rightmost. Controlling slider can suppress or amplify the different frequency parts of original music. In general there are three types of filter: low-pass, high-pass and band-pass. We will see two most popular filter in macroeconomics: the Hodrick-Prescott filter and the Baxter-King filter in this section, which are high-pass and band-pass respectively.

Filters are designed to remove trend and isolate cycles, the filtered time

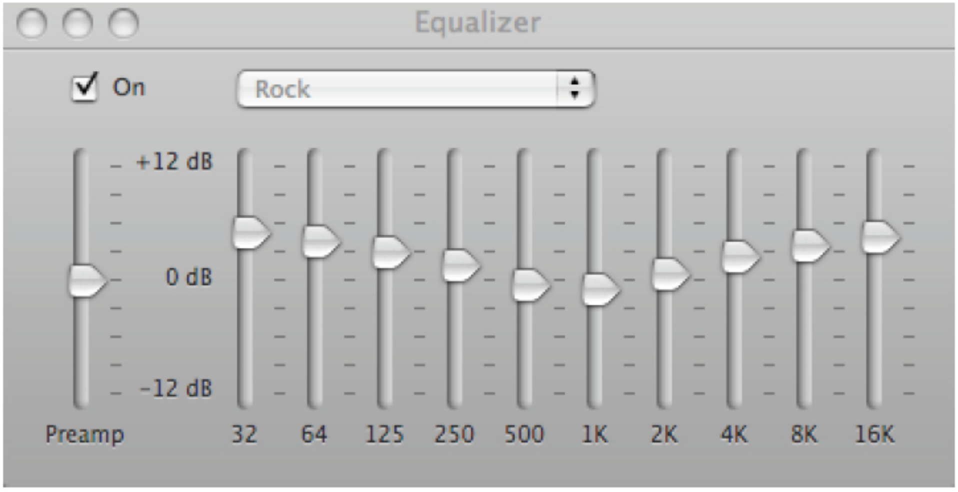

Figure 3.4: Equaliser panel

series y_t^f can be given as,

$$y_t^f = \sum_{j=-r}^{s} c_j y_{t-j} = \Big(\sum_{j=-r}^{s} c_j L^j\Big) y_t = C(L) y_t \tag{3.5}$$

where L denotes lag operator. The autocovariance between y_t^f and $y_{t-\tau}^f$ is given by

$$E\big(y_t^f y_{t-\tau}^f\big) \tag{3.6}$$

where mean-zero is assumed for y_t^f. To explore further (3.6),

$$\begin{aligned}
E\big(y_t^f y_{t-\tau}^f\big) &= E\Big(\sum_{j=-r}^{s} c_j y_{t-j}\Big)\Big(\sum_{k=-r}^{s} c_j y_{t-k-\tau}\Big) \\
&= E\sum_{j=-r}^{s}\sum_{k=-r}^{s} c_j c_k y_{y-j} y_{t-k-\tau} \\
&= \sum_{j=-r}^{s}\sum_{k=-r}^{s} c_j c_k \gamma(j-k-\tau) = \sum_{j=-r}^{s}\sum_{k=-r}^{s} c_j c_k \gamma(\tau+k-j) \\
&= \gamma_{y^f}(\tau)
\end{aligned} \tag{3.7}$$

To get power spectrum, take the Fourier transform,

$$
\begin{aligned}
s_{y^f}(\omega) &= \sum_{\tau=-\infty}^{\infty} \gamma_{y^f}(\tau) e^{-i\omega\tau} \\
&= \sum_{\tau=-\infty}^{\infty} \sum_{j=-r}^{s} \sum_{k=-r}^{s} c_j c_k \gamma(j-k-\tau) e^{-i\omega\tau}
\end{aligned}
$$

Let $h = \tau + k - j$, then

$$e^{-i\omega(h+j-k)} = e^{-i\omega h} e^{-i\omega j} e^{i\omega k}$$

Substitute back,

$$
\begin{aligned}
\sum_{\tau=-\infty}^{\infty} \sum_{j=-r}^{s} \sum_{k=-r}^{s} c_j c_k \gamma(h) e^{-i\omega h} e^{-i\omega j} e^{i\omega k} &= \sum_{j=-r}^{s} c_j e^{-i\omega j} \sum_{k=-r}^{s} c_k e^{i\omega k} \sum_{\tau=-\infty}^{\infty} \gamma(h) e^{-i\omega h} \\
&= C(e^{-i\omega}) C(e^{i\omega}) s_y(\omega) \qquad (3.8)
\end{aligned}
$$

Define a 'gain function',

$$G(\omega) = \left\| C(e^{-i\omega}) \right\|$$

It is actually a Fourier transform, because the $\gamma_{y^f}(\tau)$ is symmetric, the Fourier transform is also used a gain function here. The linear combination of complex form $e^{-i\omega}$ is still a complex form, so $\left\| C(e^{-i\omega}) \right\|$ is its modulus, which equals

$$\left\| C(e^{-i\omega}) \right\| = \sqrt{C(e^{-i\omega}) C(e^{i\omega})}$$

Then (3.8) becomes,

$$C(e^{-i\omega}) C(e^{i\omega}) s_y(\omega) = \left\| C(e^{-i\omega}) \right\|^2 s_y(\omega) = G(\omega)^2 s_y(\omega)$$

This equation shows how filters isolate cycles by attenuating or amplifying the spectrum of the original series.

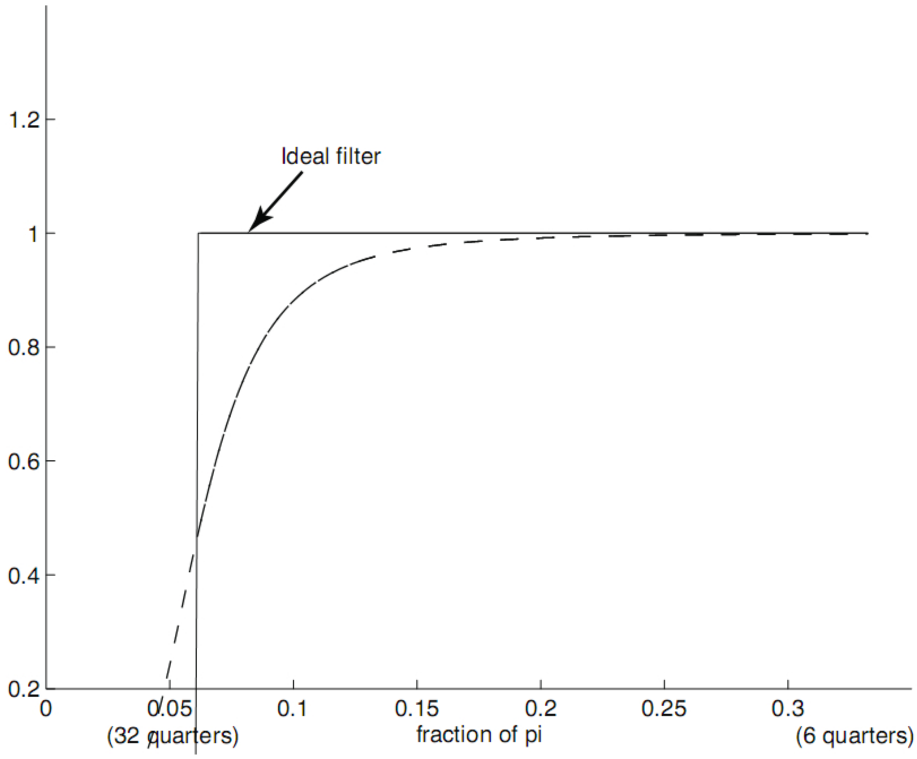

Figure 3.5: High-pass filter

3.3.1 Hodrick-Prescott Filter

The most popular detrending filter in macroeconomics is Hodrick-Prescott (H-P) filter, which decompose $\ln y_t$ as

$$\ln y_t = g_t + c_t \tag{3.9}$$

where g_t denotes the growth component, c_t the cyclical component. Filtering process is also an estimation process, which estimates g_t and c_t in order to minimise

$$\min_{\tau_t} \sum_{t=1}^{T} (y_t - g_t)^2 + \lambda \sum_{t=2}^{T-1} [(g_{t+1} - g_t) - (g_t - g_{t-1})]^2 \tag{3.10}$$

where λ is the smoothing parameter which controls the smoothness of the detrended line, as $\lambda \to 0$, the trend approximates the actual series, while

$\lambda \to \infty$ the trend becomes linear. If we rewrite (3.10),

$$\min_{\tau_t} \sum_{t=1}^{T} (y_t - g_t)^2 + \lambda \sum_{t=2}^{T-1} [\Delta g_{t-1} - \Delta g_t]^2 \tag{3.11}$$

It is clear to see that applying H-P filter involves minimising cyclical component $y_t - g_t$ and a penalty from second difference of growth component $\Delta g_{t-1} - \Delta g_t$. Hodrick and Prescott (1997) suggests to use $\lambda = 1600$. King and Rebelo (1993)[3] showed that H-P filter can render stationary of any integrated process as high as the fourth order.

The H-P filter is a typical high-pass filter which removes low frequencies, namely, the long cycle components. See figure 3.5,

The vertical axis represents squared gain, horizontal axis represents frequencies, the solid line shows the cutting line of low and high frequencies. The ideal high-pass filter is represented by solid line, the dashed line represents the HP filter.

3.3.2 The Baxter-King filter

The Baxter-King (BK) filter is an approximation of ideal band-pass (BP) filter which removes low and high frequencies. BK filter is designed to pass through 6 and 32 quarters fluctuation, which is proposed as a moving-average:

$$y_t^f = \sum_{h=-12}^{12} a_h y_{t-h} \tag{3.12}$$

where a_h is moving average weights which can be derived from the inverse Fourier transform. Because the ideal high-pass filter does not exist, it need an infinite sequence of signals, so BK filter is an approximation of BP filter.

See figure 3.6, which shows the comparison of ideal BP filter and BK filter.

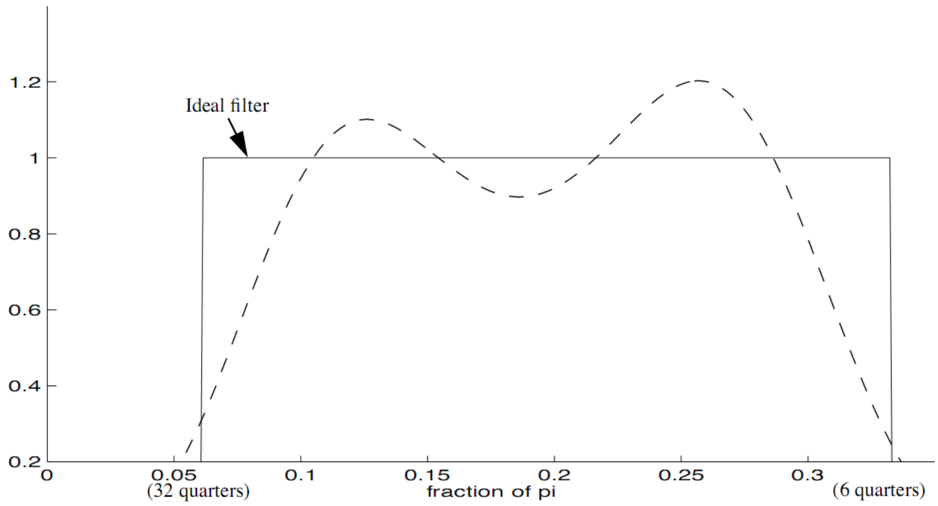

Figure 3.6: Baxter-King filter Squared Gain

Appendix A

Matlab Code for Cover Picture

```
N = 11;                           % summation limit (use N odd)
wo = pi;                          % fundamental frequency (rad/s)
c0 = 0;                           % dc bias
t = -3:0.01:3;                    % declare time values

figure(1)                         % put first two plots on figure 1

% Compute yce, the Fourier Series in complex exponential form

yce = c0*ones(size(t));           % initialize yce to c0

for n = -N:2:N,                   % loop over series index n (odd)
  cn = 2/(j*n*wo);                % Fourier Series Coefficient
  yce = yce + real(cn*exp(j*n*wo*t)); % Fourier Series computation
end
```

```
subplot(2,1,1)
plot([-3 -2 -2 -1 -1  0 0 1  1  2 2 3],...    % plot original y(t) dots
     [-1 -1  1  1 -1 -1 1 1 -1 -1 1 1], ':');
hold;
plot(t,yce,'LineWidth',3);     % plot truncated exponential FS
xlabel('t (seconds)'); ylabel('y(t)');
ttle = ['Truncated Exponential Fourier Series with N = ',...
         num2str(N)];
title(ttle);
hold;

% Compute yt, the Fourier Series in trigonometric form

yt = c0*ones(size(t));                 % initialize yt to c0

for n = 1:2:N,                         % loop over series index n (odd)
  cn = 2/(j*n*wo);                     % Fourier Series Coefficient
  yt = yt + 2*abs(cn)*cos(n*wo*t+angle(cn));  % Fourier Series computation
end

subplot(2,1,2)
plot([-3 -2 -2 -1 -1  0 0 1  1  2 2 3],...    % plot original y(t)
     [-1 -1  1  1 -1 -1 1 1 -1 -1 1 1], ':');
hold;                                   % plot truncated trigonometric FS
plot(t,yt,'LineWidth',3);
xlabel('t (seconds)'); ylabel('y(t)');
```

```
ttle = ['Truncated Trigonometric Fourier Series with N = ',...
         num2str(N)];
title(ttle);
hold;

% Draw the amplitude spectrum from exponential Fourier Series

figure(2)                          % put next plots on figure 2

subplot(2,1,1)
stem(0,c0);                        % plot c0 at nwo = 0

hold;
for n = -N:2:N,                    % loop over series index n
  cn = 2/(j*n*wo);                 % Fourier Series Coefficient
  stem(n*wo,abs(cn))               % plot |cn| vs nwo
end
for n = -N+1:2:N-1,                % loop over even series index n
  cn = 0;                          % Fourier Series Coefficient
  stem(n*wo,abs(cn));              % plot |cn| vs nwo
end

xlabel('w (rad/s)')
ylabel('|cn|')
ttle = ['Amplitude Spectrum with N = ',num2str(N)];
title(ttle);
grid;
```

```
hold;

% Draw the phase spectrum from exponential Fourier Series

subplot(2,1,2)
stem(0,angle(c0)*180/pi);           % plot angle of c0 at nwo = 0

hold;
for n = -N:2:N,                      % loop over odd series index n
  cn = 2/(j*n*wo);                   % Fourier Series Coefficient
  stem(n*wo,angle(cn)*180/pi);       % plot |cn| vs nwo
end
for n = -N+1:2:N-1,                  % loop over even series index n
  cn = 0;                            % Fourier Series Coefficient
  stem(n*wo,angle(cn)*180/pi);       % plot |cn| vs nwo
end

xlabel('w (rad/s)')
ylabel('angle(cn) (degrees)')
ttle = ['Phase Spectrum with N = ',num2str(N)];
title(ttle);
grid;
hold;
```

Bibliography

[1] Maybeck P.S. (1979): *Stochastic models, estimation and control*, Vol.1, Academic Press.

[2] Hodrick, R. and Prescott E.P. (1997): Post-war Business Cycles: An Empirical Investigation, *Journal of Money, Credit, and Banking*, 29, 1-16

[3] King, R.G. and Rebelo, S. (1993): Low Frequency Filtering and Real Business-Cycles, *Journal of Economic Dynamics and Control*