Let S_t be the stock/index price at time t

L_t be the leveraged ETF price at time t.

β be the leverage, r be the cost of leverage, f be the management fee.

We have: $\frac{dL_t}{L_t} = \beta \frac{dS_t}{S_t} + [(1-\beta)r - f]dt$. ①

with S_t satisfies $\frac{dS_t}{S_t} = \delta_t dz_t + \mu_t dt$, ② z_t is the standard Brownian Motion

μ_t is the drift coefficient and δ_t is the expansion coefficient.

According to Itô's Lemma:

$$d\ln S_t = \frac{dS_t}{S_t} - \frac{\delta_t^2}{2}dt \quad ③ \qquad d\ln L_t = \frac{dL_t}{L_t} - \frac{\beta^2\delta_t^2}{2}dt \quad ④ .$$

So $$d\ln L_t + \frac{\beta^2\delta_t^2}{2}dt = \beta\left(d\ln S_t + \frac{1}{2}\delta_t^2 dt\right) + \left((1-\beta)r - f\right)dt \quad ⑤$$

We have: $$\frac{L_t}{L_0} = \left(\frac{S_t}{S_0}\right)^\beta \exp\left\{\left((1-\beta)r - f\right)t + \frac{\beta-\beta^2}{2}\int_0^t \delta_s^2 ds\right\} . \quad ⑥$$

The decay factor is $\frac{\beta^2-\beta}{2}\int_0^t \delta_s^2 ds + (f - (1-\beta)r)t$

What we want to do is to choose leverage level β to maximize $E(\frac{L_t}{L_0})$

$$\max_\beta E\left(\frac{S_t}{S_0}\right)^\beta \exp\left\{\left((1-\beta)r - f\right)t + \frac{\beta-\beta^2}{2}\int_0^t \delta_s^2 ds\right\}.$$

Suppose $\mu_s \equiv \mu$, $\delta_s \equiv \sigma$.

$$\Leftrightarrow \max_\beta E\beta \ln\frac{S_t}{S_0} + \frac{\beta-\beta^2}{2}\int_0^t \delta_s^2 ds + \left((1-\beta)r - f\right)t .$$

$$\Leftrightarrow \max_\beta E\beta\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma z_t\right] + \frac{\beta-\beta^2}{2}\int_0^t \sigma^2 ds + \left((1-\beta)r - f\right)t .$$

$$\Leftrightarrow \max_\beta \beta\left(\mu - \frac{\sigma^2}{2}\right)t + \frac{\beta-\beta^2}{2}\int_0^t \sigma^2 ds + \left((1-\beta)r - f\right)t .$$

FOC: wrt β: $\left(\mu - \frac{\sigma^2}{2}\right)t + \left(\frac{1}{2} - \beta\right)\int_0^t \sigma^2 ds - rt = 0$.

~~We have: $\beta = \frac{(\mu - r - \frac{\sigma^2}{2})t + \frac{1}{2}\int_0^t \sigma^2 ds}{\int_0^t \sigma^2 ds}$~~

~~If σ_t~~ we have $\beta = \frac{\mu - r}{\sigma^2}$.

The leverage level ~~have~~ has no relationship with f.