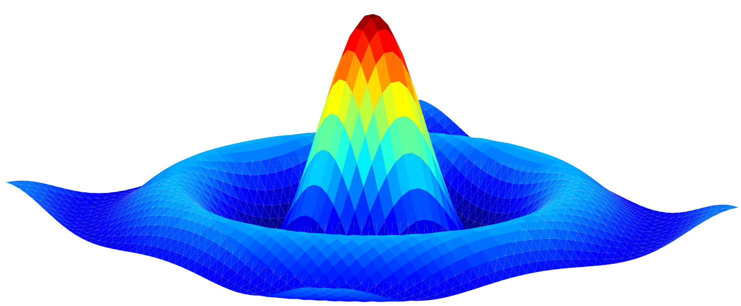
Log-linearisation Tutorial

Weijie Chen

Department of Political and Economic Studies

University of Helsinki

Updated on 5 May 2012



Abstract

To solve DSGE models, first we have to collect all expectational difference equations, such as F.O.C.s, constraints, aggregate conditions, then log-linearise them around the steady-state. It sounds easy, but it sometimes requires ingenuity and proficiency. This note is written for illustrating log-linearisation and presenting several methods, which method you are suppose to use is on your discretion.

1 Introduction

Log-linearisation, basically it means take the natural logarithm, then linearise it. Nothing mysterious, but the challenges mainly come from a technical consideration: how can we log-linearise an equation easily? 'Easily' is a relative expression, if we are about to tackle a problem, at least we are supposed to pick the right weapon.

To perform log-linearisation, we often work with log-deviation, it is defined as

$$\tilde{x}_t = \ln X_t - \ln X \tag{1}$$

where $\tilde{x}_t$ is log-deviation, X is the stead-state of X_t. To see why we need to work with log-deviation, we can manipulate equation above,

$$\tilde{x}_t = \ln\left(\frac{X_t}{X}\right) = \ln\left(1 + \frac{X_t}{X} - 1\right) = \ln\left(1 + \frac{X_t}{X} - \frac{X}{X}\right) = \ln\left(1 + \frac{X_t - X}{X}\right)$$

Now we need to employ the first-order Taylor expansion,

$$f(x_t) \approx f(x) + f'(x)(x_t - x) \tag{2}$$

where x is steady-state of x_t. Take derivative of $\ln(1 + (X_t - X)/X)$ w.r.t. X_t,

$$\frac{\partial \ln\left(1 + \frac{X_t - X}{X}\right)}{\partial X_t} = \frac{X}{X_t}\frac{1}{X} = \frac{1}{X_t}$$

Then use (2), we get

$$\tilde{x}_t = \ln\left(1 + \frac{X_t - X}{X}\right) \approx \frac{1}{X}(X_t - X) \tag{3}$$

And the right-hand side of (3) is the percentage deviation from steady state X, this is the reason we prefer to work with log-deviation. Sometimes we choose replace $X_t - X$ with $\mathrm{d}X_t$, the differential of X_t, which will be clear in later examples.

And we can also rearrange (3),

$$\tilde{x}_t \approx \frac{1}{X}(X_t - X) = \frac{X_t}{X} - 1$$
$$X_t \approx X(\tilde{x}_t + 1) \tag{4}$$

Then use this expression to substitute into all variables.

2 Universal method

Entitling this method as universal is not because it can be employed dealing with any problem, but it reveals the most fundamental idea of log-linearisation: take natural logarithm then linearise it. We shall use a Cobb-Douglas production function as the demonstrating example

$$Y_t = K^\alpha (A_t L_t)^{1-\alpha}$$

Take natural logarithm on both sides,

$$\ln Y_t = \alpha \ln K_t + (1-\alpha)\ln A_t + (1-\alpha)\ln L_t \tag{5}$$

Then we need to use Taylor expansion to the first degree around steady-state, for every terms

$$
\begin{aligned}
\ln Y_t &= \ln Y + \frac{1}{Y}(Y_t - Y) \\
\ln K_t &= \ln K + \frac{1}{K}(K_t - K) \\
\ln A_t &= \ln A + \frac{1}{A}(A_t - A) \\
\ln L_t &= \ln L + \frac{1}{L}(L_t - L)
\end{aligned}
$$

Substitute back to (5),

$$
\begin{aligned}
\ln Y + \frac{1}{Y}(Y_t - Y) = \alpha\left[\ln K + \frac{1}{K}(K_t - K)\right] + (1-\alpha)\left[\ln A + \frac{1}{A}(A_t - A)\right] \\
+ (1-\alpha)\left[\ln L + \frac{1}{L}(L_t - L)\right]
\end{aligned}
$$

Expand the equation above

$$
\begin{aligned}
\ln Y + \frac{1}{Y}(Y_t - Y) = \alpha \ln K + \alpha\frac{K_t}{K} - \alpha + (1-\alpha)\ln A + (1-\alpha)\frac{A_t}{A} \\
- (1-\alpha) + (1-\alpha)\ln L + (1-\alpha)\frac{L_t}{L} - (1-\alpha) \qquad (6)
\end{aligned}
$$

We eliminate the steady-state condition out of equation, according to (5)

$$
\ln Y = \alpha \ln K + (1-\alpha)\ln A + (1-\alpha)\ln L
$$

Then we simplified (6) a bit,

$$
\begin{aligned}
\frac{1}{Y}(Y_t - Y) = \alpha\frac{K_t}{K} - \alpha + (1-\alpha)\frac{A_t}{A} \\
- (1-\alpha) + (1-\alpha)\frac{L_t}{L} - (1-\alpha)
\end{aligned}
$$

Finally, since Y, K, A and L are only constants (parameters), we get a perfect linear function

$$\frac{Y_t}{Y} = \alpha\frac{K_t}{K} + (1-\alpha)\frac{A_t}{A} + (1-\alpha)\frac{L_t}{L} - (1-\alpha)$$

However, as you can see from steps above, the process is quite tedious, and the result is *not* log-deviation form. If the function forms are complicated, it would be sensible to turn for other methods. The rest of methods we are about to see are actually derived from this fundamental one, but they are largely simplified.

3 Total differential method

In this section, you will understand why we use $\mathrm{d}X$ to replace $X_t - X$. We use Galí and Monacelli (2005) as our example, to show you how to log-linearise the function you see in a real paper. From Galí and Monacelli (2005), we use equation pair (6) to illustrate the total differential method, reproduce them here

$$C_{H,t} = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t, \qquad C_{F,t} = \alpha\left(\frac{P_{F,t}}{P_t}\right)^{-\eta} C_t \tag{7}$$

They are optimal consumption allocation to domestic goods and foreign goods.

Take logarithm upon both sides for the first equation,

$$\ln C_{H,t} = \ln(1-\alpha) - \eta\ln P_{H,t} + \eta\ln P_t + \ln C_t$$

Then take total differential of each term on its steady-state value

$$\frac{1}{C_H}\,\mathrm{d}C_{H,t} = -\eta\frac{1}{P_H}\,\mathrm{d}P_{H,t} + \eta\frac{1}{P}\,\mathrm{d}P_t + \frac{1}{C}\,\mathrm{d}C_t$$

where C_H, P_H and P without subscripts t are steady-state value. Then rewrite them into log-deviation form

$$\tilde{c}_{H,t} = -\eta(\tilde{p}_{H,t} - \tilde{p}_t) + \tilde{c}_t \tag{8}$$

The same procedure for another equation, take logarithm on both sides

$$\ln C_{F,t} = \ln \alpha - \eta \ln P_{F,t} + \eta \ln P_t + \ln C_t$$

Take total differential on steady-state,

$$\frac{1}{C_F}\,\mathrm{d}C_{F,t} = -\eta\frac{1}{P_F}\,\mathrm{d}P_{F,t} + \eta\frac{1}{P}\,\mathrm{d}P_t + \frac{1}{C}\,\mathrm{d}C_t$$

Then log-deviation form,

$$\tilde{c}_{F,t} = -\eta(\tilde{p}_{F,t} - \tilde{p}_t) + \tilde{c}_t \tag{9}$$

This time, the linearisation goes considerably quick, and result is in log-deviation form. The Taylor expansion to the first degree is to linearise a nonlinear function, so it functions the same as total differential approximation. And obviously taking total differential is easier than Taylor expansion to the first degree, this is why we prefer not to use Taylor expansion directly.

4 Uhlig's method

In this section, we will see another interesting method proposed by Uhlig (1999), which does not even require to take derivatives. This method is just a further derivation of (1),

$$\ln X_t = \ln X + \tilde{x}_t$$

Take exponential on both sides,

$$X_t = e^{\ln X + \tilde{x}_t} = e^{\ln X} e^{\tilde{x}_t} = X e^{\tilde{x}_t}$$

Then the idea is clear, we replace every variable with its according transformed term $Xe^{\tilde{x}_t}$, where X is the steady-state value. We have several expansion rules to follow:

$$\begin{aligned} e^{\tilde{x}_t} &\approx 1 + \tilde{x}_t \\ e^{\tilde{x}_t + a\tilde{y}_t} &\approx 1 + \tilde{x}_t + a\tilde{y}_t \\ \tilde{x}_t \tilde{y}_t &\approx 0 \\ E_t[ae^{\tilde{x}_t}] &\approx E_t[a\tilde{x}_t] + a \end{aligned}$$

You get the right-hand side expression by simply taking the Taylor expansion to the first degree. But you don't need to use Taylor expansion every time, using these rules will save your lots of troubles.

One important remark is that Uhlig's method is immune to Jensen's inequality

$$\ln E_t X > E_t \ln X$$

A specific example of this inequality, we have seen in our advanced microeconomic textbook-risk aversion. Since here natural logarithm is strictly concave thus the inequality always holds. Then our problem is that we can't simply take logarithm to a function with expectation operator. One clever way to circumvent the problem is to use Uhlig's method.

The example here is still from Galí and Monacelli (2005), the equation

(10), stochastic Euler equation,

$$1 = \beta R_t E_t \left[\frac{(C_{t+1} - hC_t)^{-\sigma}}{(C_t - hC_{t-1})^{-\sigma}} \frac{P_t}{P_{t+1}} \right]$$

For sake of easy handling, we replace $\xi_t = (C_t - hC_{t-1})^{-\sigma}$,

$$1 = \beta R_t E_t \left[\frac{\xi_{t+1}}{\xi_t} \frac{P_t}{P_{t+1}} \right] \tag{10}$$

Follow method in Uhlig (1999),

$$1 = \beta R e^{\tilde{r}_t} E_t \left[\frac{\xi e^{\tilde{\xi}_{t+1}}}{\xi e^{\tilde{\xi}_t}} \frac{P e^{\tilde{p}_t}}{P e^{\tilde{p}_{t+1}}} \right]$$

Use stationary condition

$$1 = \beta R \frac{\xi}{\xi} \frac{P}{P}$$

to simplify and collecting exponential terms together,

$$1 = E_t \left[e^{\tilde{r}_t + \tilde{\xi}_{t+1} + \tilde{p}_t - \tilde{\xi}_t - \tilde{p}_{t+1}} \right]$$

Taylor expansion to the first degree on the right-hand side or simply use expansion rules,

$$1 \approx 1 + \tilde{r}_t + E_t \tilde{\xi}_{t+1} + \tilde{p}_t - \tilde{\xi}_t - E_t \tilde{p}_{t+1} \tag{11}$$

It is already linearised at this step, but we need to find out what is $\tilde{\xi}_{t+1}$ and $\tilde{\xi}_t$, we have previously defined

$$\xi_t = (C_t - hC_{t-1})^{-\sigma},$$

take logarithm on both sides,

$$\ln \xi_t = -\sigma \ln(C_t - hC_{t-1})$$

then take total differential at steady-state value,

$$\begin{aligned}
\frac{1}{\xi}\,\mathrm{d}\xi_t &= -\sigma\frac{1}{C(1-h)}[\,\mathrm{d}C_t - h\,\mathrm{d}C_{t+1}] \\
\tilde{\xi}_t &\approx -\sigma\frac{1}{C(1-h)}\left[C\frac{\mathrm{d}C_t}{C} - hC\frac{\mathrm{d}C_{t+1}}{C}\right] \\
\tilde{\xi}_t &\approx -\sigma\frac{1}{1-h}(\tilde{c}_t - h\tilde{c}_{t+1})
\end{aligned}$$

Substitute back to (11),

$$0 \approx \tilde{r}_t - E_t\left[\frac{\sigma}{1-h}(\tilde{c}_{t+1} - h\tilde{c}_t)\right] + \frac{\sigma}{1-h}(\tilde{c}_t - h\tilde{c}_{t+1}) - E_t[\tilde{p}_{t+1} - \tilde{p}_t]$$

Define $\tilde{p}_{t+1} - \tilde{p}_t = \tilde{\pi}_{t+1}$ the inflation rate, then multiply both sides by $(1-h)/\sigma$

$$E_t(\tilde{c}_{t+1} - h\tilde{c}_t) + \frac{1-h}{\sigma}E_t[\tilde{\pi}_{t+1} - \tilde{r}_t] \approx \tilde{c}_t - h\tilde{c}_{t-1} \tag{12}$$

5 Substitution method

Strictly speaking, this is not a method, it simply omits the first step of Uhlig's replacement, then directly come to (4). There is only one step needs attention, we will see in next example:

$$X_t + a = (1-b)\frac{Y_t}{L_t Z_t}$$

Taking natural logarithm on both sides,

$$\ln(X_t + a) = \ln(1-b) + \ln Y_t - \ln L_t - \ln Z_t \tag{13}$$

Use steady-state condition,

$$\ln(X + a) = \ln(1-b) + \ln Y - \ln L - \ln Z \tag{14}$$

Subtract (14) away from (13),

$$\ln(X_t + a) - \ln(X + a) = \ln Y_t - \ln Y - \ln L_t + \ln L - \ln Z_t + \ln Z$$

They all become log-deviation form

$$\widetilde{X_t + a} = \tilde{y}_t - \tilde{l}_t - \tilde{z}_t \tag{15}$$

We need to find out what $\tilde{x}_t$ is, then replace $\widetilde{X_t + a}$. Here comes the step needs attention, we use (3)

$$\tilde{x}_t \approx \frac{X_t - X}{X} \tag{16}$$

Then $\widetilde{X_t + a}$ becomes

$$\widetilde{X_t + a} \approx \frac{X_t + a - (X + a)}{X + a} = \frac{X_t - X}{X + a} \tag{17}$$

Since the numerator of right-hand sides of (16) and (17) are equal, we can make use of equality and set,

$$\begin{aligned} \widetilde{X_t + a}(X + a) &= \tilde{x}_t X \\ \widetilde{X_t + a} &= \frac{\tilde{x}_t X}{X + a} \end{aligned}$$

Substitute back to (15),

$$\frac{\tilde{x}_t X}{X + a} = \tilde{y}_t - \tilde{l}_t - \tilde{z}_t$$

This function actually can be easily linearised by total differential method, you can try by yourself, only needs two steps, everything will be done.

6 Taylor Approximation

6.1 Single variable case

We can dig further to see whether we can find some tricks to largely simplified the universal method. We will work on a general case first, a nonlinear difference equation

$$X_{t+1} = f(X_t) \tag{18}$$

where f is any nonlinear functional form you can imagine. First-order Taylor expansion of right-hand side around the steady-state is

$$X_{t+1} \approx f(X) + f'(X)(X_t - X) \tag{19}$$

If we set (18) to its steady-steady condition,

$$X = f(X)$$

Then (19) becomes

$$X_{t+1} \approx X + f'(X)(X_t - X)$$

Dividing by X,

$$\frac{X_{t+1}}{X} \approx 1 + f'(X)\frac{X_t - X}{X}$$

The left-hand side can be replaced by (4),

$$\begin{aligned} 1 + \tilde{x}_{t+1} &= 1 + f'(X)\tilde{x}_t \\ \tilde{x}_{t+1} &= f'(X)\tilde{x}_t \end{aligned} \tag{20}$$

Follow this formula, we simply take derivative at the steady-state, everything will be done.

Try an example,

$$k_{t+1} = (1-\delta)k_t + sk_t^{\alpha}$$

Use formula (20),

$$\tilde{k}_{t+1} = [s\alpha k^{\alpha-1} + (1-\delta)]\tilde{k}_t$$

It is now linearised, since the term in the square brackets are simply parameters.

6.2 Multivariable case

Taylor polynomial has a vector version as well as scalar version, actually most of functions you encounter will be multivariable rather what we had seen in last section. Again, let's start at the general case[1],

$$X_{t+1} = f(X_t, Y_t) \tag{21}$$

where f is still any nonlinear function you can imagine. The vector version (bivariate) of first-order Taylor polynomial around the steady-state is

$$X_{t+1} \approx f(X,Y) + f_X(X,Y)(X_t - X) + f_Y(X,Y)(Y_t - Y) \tag{22}$$

As you can guess, the bivariate Taylor expansion closely relates to total derivative/differential of bivariate function. Again set steady-state condition of (21): $X = f(X,Y)$, (22) becomes,

$$X_{t+1} \approx X + f_X(X,Y)(X_t - X) + f_Y(X,Y)(Y_t - Y)$$

[1] We use bivariate examples here for sake of simplicity.

Dividing by X,

$$\begin{aligned}
\frac{X_{t+1}}{X} &\approx 1 + f_X(X,Y)\frac{(X_t - X)}{X} + f_Y(X,Y)\frac{(Y_t - Y)}{X} \\
1 + \tilde{x}_{t+1} &\approx 1 + f_X(X,Y)\tilde{x}_t + f_Y(X,Y)\frac{Y}{X}\frac{(Y_t - Y)}{Y} \\
\tilde{x}_{t+1} &\approx f_X(X,Y)\tilde{x}_t + f_Y(X,Y)\frac{Y}{X}\tilde{y}_t \qquad (23)
\end{aligned}$$

(23) is the formula we are seeking for. Try an example again,

$$k_{t+1} = (1-\delta)k_t + sz_t k_t^{\alpha}$$

Then calculate partial derivatives,

$$\begin{aligned}
f_z(z,k) &= sk^{\alpha} \\
f_k(z,k) &= \alpha szk^{\alpha-1} + (1-\delta)
\end{aligned}$$

Use formula (23),

$$\tilde{k}_{t+1} = [\alpha szk^{\alpha-1} + (1-\delta)]\tilde{k}_t + sk^{\alpha}\frac{z}{k}\tilde{z}_t$$

Open the brackets,

$$\tilde{k}_{t+1} \approx \alpha szk^{\alpha-1}\tilde{k}_t + (1-\delta)\tilde{k}_t + sk^{\alpha-1}z\tilde{z}_t$$

There are still many situations we haven't covered in this note, you can find some published papers and try to log-linearise those key functions by yourself then compare the results from those provided by authors.

The End.

References

[1] McCandless, G. (2008): The ABCs of RBCs: An Introduction to Dynamic Macroeconomic Models, Harvard University Press

[2] Uhlig, H. (1999): 'A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily', *Computational Methods for the Study of Dynamic Economies*, Oxford University Press, Oxford, 30-61