Ponder This Challenge for Nov.2013

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A three-dimensional cube has eight vertices, twelve edges, and six faces. Let's call them 0-D, 1-D, and 2-D faces, respectively.

Denote f(d,k) as the number of k-dimensional faces of an d-dimensional hyper cube, so f(3,1)=12.

Find three cubes (with different dimensions d1, d2, and d3) such that the number of k1, k2, and k3 dimension faces are the same, i.e. f(d1,k1) = f(d2,k2) = f(d3,k3).

We are looking for nontrivial solutions, so k1 should be less then d1. Bonus: Find more than three.

Solutions.

We will prove $f(2^k+k-1,0) = f(2^k,1) = f(2^{2^k+k-2}, 2^{2^k+k-2}-1), \forall k \in \mathbb{N}$.

Lemma 1. $f(n,i) = \binom{n}{i}2^{n-i}$.

Proof. Consider the generating function: $g_n(x) = \sum_{i=0}^{n} f(n,i)x^i$. Notice $g_0(x) = 2+x, g_1(x) = 4+4x+x^2, \cdots, g_{k+1}(x) = (2+x)g_k(x)$, hence we have $g_n(x) = (2+x)^n$.

So compare the coefficients on both sides of the generating function, we have $f(n,i) = \binom{n}{i}2^{n-i}$. □

Corollary. Let $x = 1$ in the generating function, we have

$$1 = (2-1)^n = \sum_{i=1}^{n}(-1)^i f(n,i) = \sum_{i=1}^{n}(-1)^i \binom{n}{i} 2^{n-i}$$

When $n = 3$, the result is consist with Euler Formula in three dimension, which $V = f(3,0), E = f(3,1), F = f(3,2)$. According to **Lemma 1**, we have $V - E + F - 1 = 1$ which is exactly the Euler Formula.

Remarks. Imagine 1 as a point and x as an endge. We can see $1 + x + 1$ is the generating function in one dimension. Similarly we can get $(1 + x + 1)^n$ is the generating function in n dimensions.

Lemma 2. With $i \neq 0, 1, n-1, n$, $f(n,i)$ could not be in the form of $2^q, q \in \mathbb{N}$.

Proof. . For $n \leq 6$, we can verify **Lemma 2.** is true. Assume $n > 6$ and because of $\binom{n}{i} = \binom{n}{n-i}$, we only consider $i \leq \frac{n}{2}$.

Suppose $f(n,i) = \binom{n}{i}2^{n-i} = 2^q, q \in \mathbb{N}$, then $\binom{n}{i} = \frac{n*(n-1)*\cdots*(n-i+1)}{1*2*\cdots*i} = 2^p, p \in \mathbb{N}$. Rewrite it we have $n*(n-1)*\cdots*(n-i+1) = 2^p * 1*2*\cdots*i$. So we have

$$\begin{cases} n(n-2)\cdots(n-i+1) \mid 1*3*\cdots*i, & 2 \nmid n, 2 \nmid i \\ (n-1)(n-3)\cdots(n-i+2) \mid 1*3*\cdots*i, & 2 \mid n, 2 \nmid i \\ (n)(n-2)\cdots(n-i+2) \mid 1*3*\cdots*(i-1), & 2 \nmid n, 2 \mid i \\ (n-1)(n-3)\cdots(n-i+1) \mid 1*3*\cdots*(i-1), & 2 \mid n, 2 \mid i \end{cases}$$

Since $n > 6$ and $i \leq \frac{n}{2}$, above four relationships can never be true since

$$\begin{cases} n(n-2)\cdots(n-i+1) > 1*3*\cdots*i, & 2 \nmid n, 2 \nmid i \\ (n-1)(n-3)\cdots(n-i+2) > 1*3*\cdots*i, & 2 \mid n, 2 \nmid i \\ (n)(n-2)\cdots(n-i+2) > 1*3*\cdots*(i-1), & 2 \nmid n, 2 \mid i \\ (n-1)(n-3)\cdots(n-i+1) > 1*3*\cdots*(i-1), & 2 \mid n, 2 \mid i \end{cases}$$

□

So what we have is only the case of $i = 0, 1, n-1, n$. Because we are looking for nontrivial solutions, so $i = 0, 1, n-1$.

When $n = 0$, $f(n, 0) = 2^n$.

When $n = 1$, $f(n, 1) = n2^{n-1}$ We must have $n = 2^k, k \in \mathbb{N}$. At this time $f(n, 1) = 2^k 2^{2^k - 1} = 2^{2^k + k - 1}$. So we have

$$f(2^k + k - 1, 0) = f(2^k, 1) = f(2^{2^k + k - 2}, 2^{2^k + k - 2} - 1) = 2^{2^k + k - 1}, \forall k \in \mathbb{N}$$

□