The Derivation and Discussion of Standard Black-Scholes Formula

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In this article, we will introduce the concept of Arbitrage Pricing Theory and consequently deduce the standard option price formula. We will make some remarks on the derivation of the standard Black-Scholes Formula. Particularly we shall reiterate the importance of the continuous option duplication process, which is the key of the standard Black-Scholes Formula.

1 Arbitrage Pricing Theory

Arbitrage Pricing Theory. All the risky assets, if following the following factor structure: $r_j = a_j + \sum_{i=1}^{n} b_{ji}F_i + \epsilon_j$, where a_j is a constant for a_j, F_k is a system factor, b_{ji} is the sensitivity of the ith asset to factor j, ϵ_j is sensitivity of the th asset to factor and we have $E(\epsilon_j) = 0, \forall j$. Then the theory claims that the expected return of this asset follows

$$E(r_j) = r_f + \sum_{i=1}^{n} b_{ji} RP_i \tag{1.1}$$

where r_f is the market risk-free rate and RP_i is the risk premium of factor i

Remarks. The proof of Arbitrage Pricing Theory can be found in many textbooks. The key to the proof is the Law of One Price, which states that the portfolio with the same payoff must enjoy the same current price. Law of One price is the central theorem in modern finance theory since almost all the asset pricing theorem is based on this law. Recall the CAPM model, we can treat APT as an extension of CAPM theorem fom two dimensions to higher dimensions. Actually we can prove the APT theorem under the assumption of CAPM by using mathematical induction. Our derivation of the standard Black-Scholes Equation is based particularly on the Law of One Price and APT.

2 Itô's Lemma and Lognormal Distribution

In the history of stock market research, what concerns us the most is the the distribution of the stock price. In the research history scholars have raised a lot of models of the rate of return of stock. Here we will briefly discuss three kinds of rate of return distribution.

2.1 Normal Distribution

At the very beginning period of financial market research, we assume the simple rate of return $\{R_t | t = 1, \cdots, T\} \sim$ i.i.d.$N(\mu, \sigma^2)$,where μ and σ are both constant numbers. This assumption is based on the Central Limit Theorem and meanwhile the normal distribution is easy to be handled with in both empirical and theoretical research.

However, this assumption has several severe problems which can never be overcomed so that recent papers stop using this assumption. Firstly, $\prod_{t=1}^{k} R_t$ is should be the multiperiod rate of return of stock, which ought to follow the normal distribution. However, the production of k i.i.d. normal distribution is no longer a normal distribution. Secondly, $R_t \geq -1$ is true for any simple rate of return, which contradicts that normal distribution has no lower bound. Thirdly and most importantly, the normal distribution assumption contradicts the empirical research results in recent years. Intuitionly, the real stock market is so complicated that we can never use such simple asumptions to simulate the true stock market.

2.2 Lognormal Distribution

To solve the problem that the rate of return has a lower bound -1 and the product of multiperiod distribution should follow the same distribution, we shall assume that the rate of return series follows lognormal distribution. In other words, we may assume that the logarithm series of rate of return follows the normal distribution.

Different from the normal distribution assumption, lognormal distribution can overcome the two of three abovementioned weakness. First, the lognormal ditribution has a range from $-\infty$ to ∞ which is exactly the accessible range of logarithm of rate of return. Second, the summation of the lognormal distribution is also a lognormal distribution, which is exactly the multiperiod lognormal distribution. We can sum up the log-return of some single periods to get a multiperiod log-return. In this meaning, this assumption provides us an easy way to deal with the original datas. Consequently, the standard Black-Scholes Formula requests the return of stock follows lognormal distribution. In the following article, we shall use Itô's formula, APT and lognormal distribution to derive the standard European put option price formula.

But we still want to mention some weakness of the lognormal distribution. Empirical study shows many stocks had excess kurtoness which the lognormal distribution fails to provide. That is to say, many stocks exhibit fat tail characteristics that is different from normal distribution, particularly in 1987 Stock Market Disaster and 2008 Global Financial Crisis. The fat tail characteristics greatly affect the option pricing since the investors may lose huge money under extreme case which can actually happen while many options expand this kind of loss by offering huge leverage. Under the lognormal distribution assumption, we should offer risk premium to the standard Black-Scholes option price to compensate for the protential fat tail great loss.

2.3 Stable Distribution

In order to overcome the weakness of lognormal distribution, the statisticians provide us a different kind of distribution called Stable Distribution(also called Lévy α-stable Distribution) to overcome the lack of excess skewness, which is developed by Lévy, a famous French mathematician.

The density function of Stable distribution is defined by its characteristic function.

$$f(x;\alpha,\beta,c,\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t)e^{-itx}dt, \tag{2.1}$$

where

$$\phi(t) = \exp(it\mu - |ct|^{\alpha}(1 - i\beta sgn(t)\Phi)) \tag{2.2}$$

where

$$sgn(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases} \tag{2.3}$$

and

$$\Phi = \begin{cases} \tan(\frac{\pi\alpha}{2}), & \alpha \neq 1 \\ -(\frac{2}{\pi}) \log |t|, & \alpha = 1 \end{cases} \tag{2.4}$$

We shall briefly explain the meaning of the parameters in the defition of stable distribution. $\mu \in \mathbb{R}$ is a shift parameter while $\beta \in [-1, 1]$ is a skewness parameter.Notice under the case $\beta = 0$, the characteristic function is just a stretched exponential function. $\alpha \in (0, 2]$ is an index to evaluate the excess kurtosis, which is really useful to depict the real financial market. The parameter $|c| > 0$ is a scale factor.

We also expect the stable distribution has the characteristics that the summation of several stable distribution should also be a stable distribution. Fortunately the stable distribution satisfies this requirement. Let $X_i \sim f(x; \alpha, \beta, c, \mu)$ and $Y = \sum_{i=1}^{N} k_i(X_i - \mu)$, then $Y \sim \frac{1}{s} f(y/s; \alpha, \beta, c, 0)$ where $s = (\sum_{i=1}^{N} |k_i|^{\alpha})^{\frac{1}{\alpha}}$.

Stable distribution can provide us great characteristics such as skewness and excess kurtosis which is more correspondent with the real financial market. But it also bring us difficultiesin model construction and calculation. We would rather not use this distribution, for under many cases we can never transform the equation into a standard heat equation to give an expression of option price.

2.4 Itô's Lemma and Stock Price

Here we assume the simple rate of return of a stock follows lognormal distribution for the abovementioned reasons and deduce the stochastic differential equation by using Itô's Lemma. Let $S(t)$ be the stock price a time t, according to the lognormal distribution, $S(t)$ satisfies a stochastic differential equation.

$$dS = \mu S dt + \sigma S dB_t \tag{2.5}$$

where B_t represents standard Brownian Motion μ is the drift parameter and σ is the diffusion parameter. We can rewrite this equation as

$$dS/S = \mu dt + \sigma dB_t \tag{2.6}$$

It makes clearer $S(t)$ follows lognormal distribution. We will solve the stock price by using Itô's Lemma.

Remarks. Brownian Motion is initially been used to decribe the pollen motion in physics. Later with the development of financial derivative market, Brownian Motion has been used in financial market to depict the path of stocks, derivatives or many other financial derivatives.

Itô's Lemma. For an Itô drift-diffusion process

$$dX_t = \mu_t dt + \sigma_t S dB_t \tag{2.7}$$

We have

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} dB_t \tag{2.8}$$

With *Itô's Lemma*, we have

$$d \ln S = \frac{dS}{S} - \frac{1}{2S^2} dSdS = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t \tag{2.9}$$

Integrate (2.9) from 0 to t, we get

$$S(t) = S(0) \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\right) \tag{2.10}$$

Remarks. If we simply assume stock price follows normal distribution rather than standard normal distribution. We shall have

$$dS = \mu dt + \sigma dB_t \tag{2.11}$$

In this case, when we integrate (2.11), we shall have the following result under the normal distribution assumption.

$$S(t) = S(0) + \mu t + \sigma B_t \tag{2.12}$$

3 The Derivation of Black Scholes Equation

Let $V(S,t)$ be the price of European call option at time t with respect to the stock price S. We will built a portfolio of purchasing x share at the same time we sell one Eurpean put option. Then this portfolio Π can be written as

$$\Pi = xS - V \tag{3.1}$$

We should carefully select the number x to hedge the risk of this portfolio. That is to say, no matter how stock price changes, the value of this portfolio remains constant at time t. That is to say

$$\frac{\partial \Pi}{\partial S} = 0 \tag{3.2}$$

From (3.2) we can solve $x = \frac{\partial V}{\partial S}$.

Remarks. If we adjust the number of stocks continuously to hedge the risk of the portfolio, Π will have no risk at all at any time. But this kind of reproduction adjustment requires the standard option pricing formula, which has not yet been deduced. But now with computer science technology, we can accomplish such kind of reproduction in any moment.But the existence of reproduction errors and transaction fees make this kind of precise reprodution absolutely impossible. We will discuss the dilemma of continuous reproduction and the transaction fee later, which is a systematic error in the application of standard Black-Scholes Formula.

We find that portfolio Π has no risk locally so that it can be treated as a kind of risk free asset. According to the APT in the first chapter, we can easily deduce the price of Π by treating it as a risk free asset. That is to say

$$d\Pi = r\Pi dt \tag{3.3}$$

Then we differentiate Π with respect to t

$$d\Pi = xdS - dV \tag{3.4}$$

From *Itô's Lemma*, we can deduce

$$\begin{aligned} dV &= \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2 \\ &= \frac{\partial V}{\partial S} dS + (\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}) dt \end{aligned} \tag{3.5}$$

Hence we get

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}) dt \tag{3.6}$$

We plugin (3.6) into (3.3) and get

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = -r(\frac{\partial V}{\partial S} S - V) \tag{3.7}$$

We simplify it and then get the famous Black-Scholes Equation

$$\frac{\partial V}{\partial t} = rV - rS \frac{\partial V}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \tag{3.8}$$

with the boundary condition

$$V(S, T) = \max\{S(T) - K, 0\} \tag{3.9}$$

where $S(T)$ is the stock price at time T and K is the strike price of the option.

Remarks. If we assume the stock price follows normal distribution rather than the lognormal distribution, we will get the rewrite (3.6) as

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2}) dt \tag{3.10}$$

And thus we can get the option price follows

$$\frac{\partial V}{\partial t} = rV - rS \frac{\partial V}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \tag{3.11}$$

Later on we will find this equation can not be tranformed into a standard heat function or any other standard form of partial differential equations and therefore has no explicit solution. Actually what impressed us the most is that Black-Scholes Equation has an explicit solution that can be written in two lines clearly and accurately, which help Scholes and Merton win Nobel Prize in Economics. What a good luck of the lognormal distribution assumption! It provides us both rationary with no lower boundary and solvability with an explicit solution.

4 The Solution of Black-Scholes Equation

We want to transform (3.8) into a traditional heat function by parameter replacement in this chapter.

Notice that

$$\frac{\partial(e^{r(T-t)} V)}{\partial t} = e^{r(T-t)} (\frac{\partial V}{\partial t} - rV) \tag{4.1}$$

We replace V with $V(S, t) = e^{-r(T-t)} V'(S, t)$

$$\frac{\partial V'}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V'}{\partial S^2} + rS \frac{\partial V'}{\partial S} = 0 \tag{4.2}$$

Let $\tau = T - t$, then we get

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} + rS\frac{\partial W}{\partial S} \tag{4.3}$$

which $W(S,\tau) = V'(S, T-\tau)$

Replace $S = e^{\xi}$, we have

$$\frac{\partial W}{\partial \xi} = S\frac{\partial W}{\partial S}, \ \frac{\partial^2 W}{\partial \xi^2} = S\frac{\partial W}{\partial S} + S^2\frac{\partial^2 W}{\partial S^2} \tag{4.4}$$

Then we get

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial \xi^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial W}{\partial \xi} \tag{4.5}$$

Again let $x = \frac{1}{\sigma}(\xi + (r - \frac{1}{2}\sigma^2)\tau)$, let $U(x,\tau) = W(e^{\xi}, \tau)$, we have

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\frac{\partial^2 U}{\partial x^2} \tag{4.6}$$

which is a standard heat equation.

Notice that

$$\begin{aligned} V(S,t) &= e^{-r(T-t)}V'(S,t) = e^{-r(T-t)}W(e^{\log S}, T-t) \\ &= e^{-r(T-t)}U(\log S/\sigma + \frac{1}{\sigma}(r - \frac{1}{2})\sigma^2(T-t), T-t) \end{aligned} \tag{4.7}$$

Notice the standard solution of heat equation under $U(x,0) = f(x)$ is

$$U(x,\tau) = \frac{1}{\sqrt{2\pi\tau}}\int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{2\tau}\right) f(y)dy \tag{4.8}$$

Suppose $V(S,T) = v(S)$, then

$$U(x,0) = V(e^{\sigma x}, T) = v(e^{\sigma x}) \tag{4.9}$$

Hence we get

$$V(S,t) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_{0}^{+\infty} \exp\left(-\frac{(\log S + (r - \frac{1}{2}\sigma^2)\tau - y)^2}{2\tau\ \sigma^2}\right) v(e^y)dy \tag{4.10}$$

where $\tau = T - t$ Let $u = e^y$, we have

$$V(S,t) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_{0}^{+\infty} \exp\left(-\frac{(\log \frac{S}{u} + (r - \frac{1}{2}\sigma^2)\tau)^2}{2\tau\ \sigma^2}\right) v(u)\frac{du}{u} \tag{4.11}$$

If it is a European call option, we have $v(u) = \max\{u - K, 0\}$, then

$$V(S,t) = S\Phi(d) - Ke^{-r\tau}\Phi(d - \sigma\sqrt{\tau}) \tag{4.12}$$

where

$$\Phi(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{t^2}{2}}dt \tag{4.13}$$

is the cummulative distribution function of the standard normal distribution.

And

$$d = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \tag{4.14}$$

Corollary. With repect to the European put option, we have

$$v(u) = \max\{K - u, 0\} \tag{4.15}$$

Then

$$V(S,t) = -S(1 - \Phi(d)) + Ke^{-r\tau}(1 - \Phi(d - \sigma\sqrt{\tau})) \tag{4.16}$$

Remarks. We successfully tranform the Black-Scholes Equation into a traditional heat function. It is common sense that heat function could be solved by Fourier Transformation. So we get the explicit solution of an European call option. Intuitionally, this solution is really beautiful, since it follows our initial intuition that the option price should be current stock price with an coefficient minus strike price with an coeffient. And with time goes to the maturity, the option price should approach to $S - K$.

5 Discussions

5.1 The pricing of American call option

American call option is a kind of option that can be exercised at any time. Different from European option, American option provides you more oppotunity and freedom to exercise. At the first glance, we may find the price of American option should be high than the same European option. To be more precisely and more carefully, we can only make the conclusion that the price of an American option should be no less than a European option with same terms on the contract. Later on we will find this carefulness is really essential because we will demonstrate that the price of American option is exactly equivalent to the same European option.

Optional stopping theorem. Let $X = (X_t)_{t\in\mathbb{N}_0}$ be a discrete-time martingale and τ a stopping time with values in $\mathbb{N}_0 \cup \infty$, both with respect to a filtration (F_t). Assume that one of the following three conditions holds:

(a) The stopping time τ is almost surely bounded, i.e., there exists a constant $c \in \mathbb{N}$ such that $\tau \leq c$ a.s.

(b) The stopping time τ has finite expectation and the conditional expectations of the absolute value of the martingale increments are almost surely bounded, more precisely, $\mathbb{E}[\tau] < \infty$ and there exists a constant c such that $\mathbb{E}\left[|X_{t+1} - X_t||\mathscr{F}_t\right] \leq c$ almost surely on the event $\{\tau > t\}$ for all $t \in \mathbb{N}_0$.

(c) There exists a constant c such that $|X_{t\wedge\tau|} \leq c$ a.s. for all $t \in \mathbb{N}_0$.

Then X_τ is an almost surely well defined random variable and $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$.

Similarly, if the stochastic process X is a submartingale or a supermartingale and one of the above conditions holds, then

$$\mathbb{E}[X_\tau] > \mathbb{E}[X_0], \tag{5.1}$$

for a submartingale, and

$$\mathbb{E}[X_\tau] < \mathbb{E}[X_0] \tag{5.2}$$

for a supermartingale.

Suppose the investor chooses at time σ to exercise an American option, which σ is a stopping because it only relies on the information ahead of time σ. So we have

$$V_0(\sigma) = \hat{\mathbb{E}}[(1+r)^{-\sigma} \max\{S_\sigma - K, 0\}] \tag{5.3}$$

Therefore the American option price should be

$$V_0 = \max_{\sigma} V_0(\sigma) \tag{5.4}$$

Obviously we have

$$V_0(\sigma) \geq (1+r)^{-T}\hat{\mathbb{E}}[\max\{S_\sigma - K, 0\}] \tag{5.5}$$

We will prove

$$V_0(\sigma) \leq (1+r)^{-T}\hat{\mathbb{E}}[\max\{S_\sigma - K, 0\}] \tag{5.6}$$

Notice that $\{(1+r)^{-n}S_n\}$ is martingale with respect to $\mathbb{P}$, we have

$$\hat{\mathbb{E}}[(1+r)^{-n}(S_n - K)|\mathscr{F}_{n-1}] \geq (1+r)^{n-1}(S_{n-1} - K) \tag{5.7}$$

Therefore

$$\hat{\mathbb{E}}[(1+r)^{-n}\max(S_n - K, 0)|\mathscr{F}_{n-1}] \geq (1+r)^{n-1}\max(S_{n-1} - K, 0) \tag{5.8}$$

We have yet proved that $\{\hat{\mathbb{E}}[(1+r)^{-n}\max(S_n - K, 0)\}$ is a submartingale. By *Optional stopping theorem* we have

$$V_0(\sigma) \leq (1+r)^{-T}\hat{\mathbb{E}}[\max\{S_\sigma - K, 0\}] \tag{5.9}$$

Hence we finally get

$$V_0(\sigma) = (1+r)^{-T}\hat{\mathbb{E}}[\max\{S_\sigma - K, 0\}] \tag{5.10}$$

That is to say, the price of Amerian call option is just the same of European option.

Remarks. It seems very strange that the American option does not worth much than the European option. What we prove above is just it is unwise to exercise the American call option before the end the contract. We are expected to earn much if we keep holding the American option rather than exercising it before its end in any cases, which indicates us the intrinsic value of American option does not outweigh the European option.

5.2 The Reproduction Process Discussion

The key point is the deduction of Standard Black Scholes Equation is we should keep the $\frac{\partial \Pi}{\partial S} = 0$ at all time, which is actually a Delta-Neutral strategy. But we have to keep our share holding $x = \frac{\partial V}{\partial S}$ at any time. But it might cause heavy transaction fee while we keep a Delta-Neutral stragety. The following is a simple illustration. From the derivation of Black-Scholes Equation, i.e. (3.6) and (3.3), we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r\Pi = const. \tag{5.11}$$

which $\frac{\partial V}{\partial t}$ represents the insurance utility of an option(the more fluctuation, the higher the insurance utility of an option) while $\frac{\partial^2 V}{\partial S^2}$ represents the frequency we should adjust our portfolio to keep the delta zero. Notice They are negative correlated with each other, which means the value of the existence of an option should grealty relies on the frequent adjustment of the share quantities. Therefore, in order to realize the risk-averse utility of the option, we have to spend a lot of transaction fee to adjust our portfolio dynamically, which is a dilemma.

Remarks. The Black-Scholes Equation gives us an explicit expression of a European call option price under the assumption of lognormal distribution and no transaction fee. But we have proved in order to hold (3.8), we have to dynamically adjust our share quantities which can cause huge transaction fee. It is a weakness of Black-Scholes Equation in the real market which can not be overcomed. It is similar to the friction-free assumption in Newton's Law. However, the difference is that we can add the factor of friction to the computation so that the original form of Newton's Law will not be changed. While if we add the transaction fee to our portfolio, the original Black-Scholes Equation can not hold any more.